Abstract

These are my course notes taken in Winter 2014 as part of PMATH 370, Chaos and Fractals, a pure math course taught at University of Waterloo by Professor Kevin Hare. Credit to Miguel Wong for LaTex template.

Topics Covered

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1 Iterates of Functions

1.1 Iterates, fixed and period points

A key concept in Chaos Theory and Fractals is iterated functions - functions applied repeatedly over and over to itself. The pattern that will arise from this process will depend on the initial value from which we iterate and certain initial values have particular properties of interest.

Definition 1.1 (Iterate)**.** *Let $f : D \to D$ and $x_0 \in D$. Then $x_1 = f(x_0)$ is the first iterate, $x_2 = f(x_1) = f(f(x_0)) = f^{[2]}(x_0)$ is the second iterate and in general, $x_n = f(x_{n-1}) = f(f(...f(x_0)...)) = f^{[n]}(x_0)$ is the nth iterate of x_0.*

Definition 1.2 (Orbit)**.** *The orbit of x_0 under $f(x)$ is $\{x_0, x_1, x_2, ...\}$*

The famous Collatz conjecture involves an iterated function. Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} n/2 & \text{if n is even} \\ 3n+1 & \text{if n is odd} \end{cases}$$

The orbit of 17 under f is, for example, $\{17, 52, 13, 40, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, ...\}$. #1

In general, whether the orbit any x_0 will eventually end in a repetition of 4, 2, 1, 4, 2 ... is unknown and is a famous conjecture.

Applications of Newton's methods involve iterated functions. Consider, for example, Newton's method on $\sin(x)$, using $f(x) = x - \frac{\sin(x)}{\cos(x)}$.

The orbit of 1 is $\{1, -0.56, 0.065, -0.000095, ...0, 0, ...\}$ #2
The orbit of 2 is $\{2, 4.2, 2.5, 3.3, 3.1, ..., \pi, \pi, ...\}$

As expected, since $0, \pi$ are roots of $\sin(x)$.

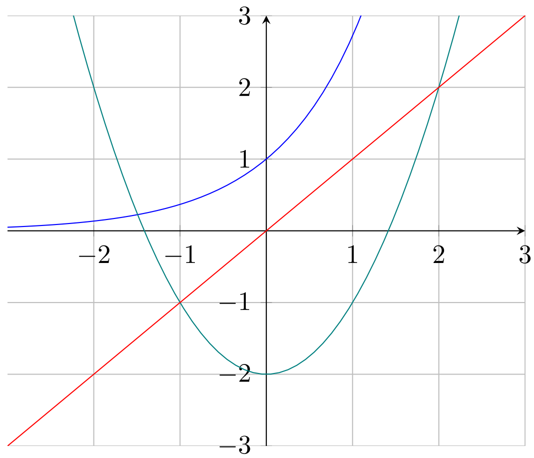
Definition 1.3 (Fixed point)**.** *Let $f : D \to D$. We say $p \in D$ is a fixed point of $f(x)$ if $f(p) = p$.*

From this definition, we see that the orbit of a fixed point is just a repetition of the fixed point. To find fixed points, then, it suffices to solve $f(p) = p$.

Find all fixed points points of $f(x) = x^2 - 2$.

Solving $f(x) = x$, we get $x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0$ so the fixed points are {-1, 2}.

On the other hand, e^x has no fixed points.



#3

Theorem 1.1. *Let $f(x)$ be continuous at p and let $\lim_{n\to\infty} f^{[n]}(x_0) = p$ for some x_0. Then p is a fixed point of $f(x)$.*

Proof. Let $x_n = f(x_{n-1}) = f^{[n]}(x_0)$. We know $\lim_{n\to\infty} x_n = p$ by definition and that $f(x)$ is continuous at p. Hence, $\lim_{n\to\infty} f(x_n) = f(p)$. We also have that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = p$. Since limits are unique, $f(p) = p$, so p is a fixed point.

□

Definition 1.4 (Attractive point)**.** *A fixed point p is* **attractive** *if there exists an interval containing p, say $I = [p-\epsilon, p+\epsilon]$, such that $\forall x_0 \in I$, $\lim_{n\to\infty} f(x_n) = p$.*

$f(x) = x^2$ has an attractive fixed point at $x = 0$, since whenever $|x| < 1$, the iterates of x get smaller and smaller.

#4

Definition 1.5 (Repelling point)**.** *A fixed point p is* **repelling** *if there exists an interval containing p, say $I = [p-\epsilon, p+\epsilon]$, such that $\forall x_0 \in I, x_0 \neq p$, we have $|f(x_0) - p| > |x_0 - p|$.*

$f(x) = \sqrt[3]{x}$ has an repelling fixed point at $x = 0$, since whenever $|x| < 1$, the iterates of x get larger and larger.

#5

Theorem 1.2. *Let p be a fixed point of $f(x)$ and assume $f(x)$ is differentiable at p. Then*

1) If $|f'(p)| < 1$, then p is attractive.
2) If $|f'(p)| > 1$, then p is repelling.

Proof. Let p be a fixed point and $|f'(p)| < 1$. Then $\lim_{x\to p} |\frac{f(x)-f(p)}{x-p}| < 1$ by the limit definition of the derivative. Let A be some value such that $|f'(p)| < A < 1$.

There will exist an interval I around p such that for all values of x in I, $|\frac{f(x)-f(p)}{x-p}| < A \implies |f(x) - p| < A|x - p|$. Since $A < 1$, this equation tells us that $x_1 = f(x)$ is closer to p than $x_0 = x$, so x_1 is in I also.

Similarly, $|x_2 - p| < A|x_1 - p| < A^2|x_0 - p|$ and in general, $|x_n - p| < A^n|x_0 - p|$. Since $A^n \to 0$, we have that $\lim_{n\to\infty} |x_n - p| = 0$.

$\therefore p$ is an attractive fixed point.

The proof for $|f'(p)| > 1$ follows the same logic.

□

The previous theorem has applications to Newton's method. Let $f(x)$ be a continuous, twice differentiable function with a simple root at p (i.e., $f(p) = 0$ and $f'(p) \neq 0$. Let $g(x) = x - f(x)/f'(x)$ be Newton's iterate. Then p is an attractive fixed point.

Proof. First we confirm that p is a fixed point by observing that $g(p) = p - f(p)/f'(p) = p - 0/f'(p) = p$.

Next, $g'(p) = 1 - \frac{f'(x)f'(x)-f''(x)f(x)}{(f'(x))^2} \implies g'(p) = 1 - \frac{(f'(p))^2-f''(p)f(p)}{(f'(p))^2} = 1 - \frac{(f'(p))^2}{(f'(p))^2} = 1 - 1 = 0$ since $f(p) = 0$. #6

$\therefore |g'(p)| = 0 < 1$, so p is an attractive fixed point by the previous theorem.

□

Definition 1.6 (Periodic points)**.** *A point p is a periodic point of period n if*

1) $f^{[n]}(p) = p$
2) $f^{[m]}(p) \neq p$ for $1 \leq m < n$

We call $x_0, x_1, ..., x_n$ the n-cycle associated to the periodic point.

The function $f(x) = -\sqrt[3]{x}$ has periodic points of period 2 at $x = 1$ and $x = -1$ with 2-cycle {1, -1} and a period point of period 1 (i.e., a fixed point) at $x = 0$ with a 1-cycle 0.

This is everything. To see this, let $0 < |x_0| < 1$. Then $0 < |x_0| < |x_1| < ... < |x_n| < 1$ for all n, so x_0 is never a period-n point. #7

Similarly, if $|x_0| > 1$, then $|x_0| > |x_1| > ... > |x_n| > 1$ for all n so again, $|x_0|$ is not a periodic point.

$\therefore$ there are no other periodic points.

Definition 1.7 (Eventually period/fixed)**.** *We say a point p is eventually periodic or fixed if there exists an m such that $f^{[m]}(p)$ is periodic or fixed.*

With $f(x) = x^2$, both 0 and 1 are fixed points and -1 is an eventually fixed point (take $m = 1$). Note that 0 is attractive and 1 is repelling since $f'(x) = 2x$. #8

There is a relationship between period points and fixed points. Namely, if p is a periodic point of period n of $f(x)$, then p is a fixed point of $f^{[n]}(x)$. Using this observation, the notion of attractive and repelling points can be extended naturally to periodic points.

Definition 1.8. *We say a periodic point of $f(x)$ is attractive or repelling if p is an attractive or repelling fixed point of $f^{[n]}(x)$.*

Theorem 1.3. *If $f^{[n]}(x)$ is differentiable at p, a periodic point of period n, then :*

1) If $|f^{[n]'}(p)| < 1$, then p is attractive.
2) If $|f^{[n]'}(p)| > 1$, then p is repelling.

Proof. Same as theorem 1.2. □

Let $f(x) = x^2 + x - 2$. Find and classify its fixed points and periodic points of period 2.

To find the fixed points, solve $f(x) = x$ which gives $x = \pm\sqrt{2}$. Furthermore, $f'(x) = 2x + 1$ which means that both fixed points are repelling since $|f'(\pm\sqrt{2})| > 1$.

To find periodic points of period 2, we need to find fixed points of $f(f(x)) = x^4 + 2x^3 - 2x^2 - 3x$. Solving $f(f(x)) = x$ gives $x(x^2 - 2)(x + 2) = 0$, which has four roots : $\sqrt{2}, -\sqrt{2}, 0, -2$. The former two are known to be fixed points of $f(x)$, so they not do fit the definition of periodic points of period 2. #9

Hence, we have that 0, -2 are periodic points of period 2.

Theorem 1.4. *For an n-cycle $\{x_0, x_1, ..., x_n\}$ with $f^{[n]}(x_0) = x_0$, $f^{[n]'}(x_0) = f^{[n]'}(x_1) = f^{[n]'}(x_2) = ... = f^{[n]'}(x_{n-1})$.*

Proof. Exercise (use chain rule a lot). □

Theorem 1.5 (Fake period m points)**.** *If $n|m$ and $T^{[n]}(p) = p$, then $T^{[m]}(p) = p$.*

Proof. Let $k = \frac{m}{n}$. To prove this, simple observe that $T^{[m]}(x)$ is the kth iterate of $T^{[n]}(x)$. That is

$$T^{[m]}(p) = \underbrace{T^{[n]}(T^{[n]}(...T^{[n]}(T^{[n]}}_{k}(p))...)) = T^{[n]}(T^{[n]}(...T^{[n]}(p)...)) = \ldots = T^{[n]}(p) = p$$

□

1.2 Families of functions

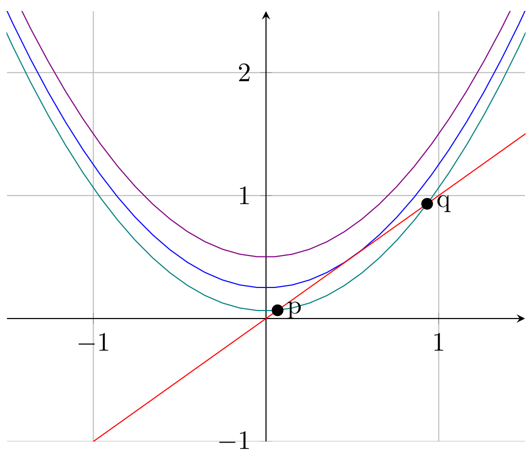
In this section, we use the concepts from the previous section to study four families of function over a parameter μ and how μ affects the existence, number, type of fixed and periodic points.

1.2.1 The family $\mathbf{g}_\mu(\mathbf{x}) = \mathbf{x}^2 + \mu$ (quadratic equation)

Consider the function $g_\mu(x) = x^2 + \mu$. Does this have any fixed points or periodic points?

To find fixed points, solve $g_\mu(x) = x \implies x^2 - x + \mu = 0 \implies x = \frac{1 \pm \sqrt{1-4\mu}}{2}$.

If $1 - 4\mu > 0$, the function has 2 fixed points (teal).
If $1 - 4\mu = 0$, the function has 1 fixed point (blue).
Otherwise, the function has no fixed points (violet).



With $\mu < 1/4$, the two fixed points are $p = \frac{1-\sqrt{1-4\mu}}{2}$ and $q = \frac{1+\sqrt{1-4\mu}}{2}$. Notice that $g_\mu'(x) = 2x$ so $g_\mu'(p) = 1 - \sqrt{1-4\mu} < 1$ and $g_\mu'(q) = 1 + \sqrt{1-4\mu} > 1$ so p is an attractive point and q is a repelling point.

Note that $g_\mu(-x) = g_\mu(x)$ so we know about the negative values of x by symmetry.

Case 1 : $x \in [0, p)$

In this interval, $x < g_\mu(x) < p$ so $g_\mu(x) \in [0, p)$ also. Hence we get that $0 \le x < g_\mu(x) < g_\mu^{[2]}(x) < \ldots < p$ which gives an increasing monotonic and bounded sequence. By the monotone convergence theorem, the limit exists - i.e., $\lim_{n\to\infty} g_\mu^{[n]}(x) = L$ for some L.

By previous theorem 1.1, L must be a fixed point $\le p$. Since the only other fixed point of $g_\mu(x)$ is $q > p$, $L = p$ and $\forall x \in [0, p)$, we have $\lim_{n\to\infty} g_\mu^{[n]}(x) = p$.

Case 2 : $x \in (p, q)$

In this interval, $p < g_\mu(x) < x$ and by a similar argument as Case 1, we get a decreasing sequence whose limit is p.

$$q > x > g_\mu(x) > g_\mu^{[2]}(x) > \ldots > p$$

Case 3 : $x > q$

Since $g_\mu(x) > x \ \forall x > q$, we see that $q < x < g_\mu(x) < g_\mu^{[2]}(x) < ....$ This sequence cannot be bounded because there are not fixed points greater than q.

From these three cases, the observation is that every point either goes to p or to infinity. That is, the iterates of every point are either increasing or decreasing except for the fixed points. Thus, points do not repeat, so there are not period points.

When $\mu = 1/4$, p and q become a single fixed point at $1/2$ and $g'_\mu(p) = 1$. This is our first example of a fixed point that is neither attractive or repelling - it attracts points to its left and repels points to its right. Finally, when $\mu > 1/4$, every point goes to infinity.

1.2.2 The family $\mathbf{Q}_\mu(\mathbf{x}) = \mu\mathbf{x}(\mathbf{1} - \mathbf{x})$ (logistic equation)

We can model population at discrete time steps with $N_{n+1} = \mu N_n(1 - N_n)$ where μ represents the birth rate (sometimes combined with the death rate), N_n is the current population such that a smaller value in the present means a smaller value in the future and $1 - N_n$ penalizes population growth when the population is too large and resources become scarce.

Let $Q_\mu(x) = \mu x(1 - x)$ be the function representing this model. As we want $Q_\mu : [0, 1] \to [0, 1]$, we restrict our analysis to $0 \le \mu \le 4, 0 \le x \le 1$.

See `https://www.desmos.com/calculator/8oinxdhcsc` for an interactive version of this function.

Case 1 : $0 \le \mu < 1$

$Q_\mu(x)$ has an attractive fixed point at $x = 0$. When solving for x in $Q_\mu(x) = \mu x(1 - x) = x$, we also notice that there is a fixed point at $x = 1 - \frac{1}{\mu}$. However, since it is negative, it is outside of our domain of interest.

$Q'_\mu(x) = \mu(1 - x) - \mu x = \mu - 2\mu x$ is equal to μ at $x = 0$. Since $\mu < 1$ by assumption, $x = 0$ is an attractive fixed point.

Case 2 : $\mu = 1$

The fixed point $x = 0$ of $Q_\mu(x)$ is attractive [*exercise*].

Case 3 : $1 < \mu < 3$

Again, $Q_\mu(x)$ has two fixed points at $x = 0$ and $x = 1 - \frac{1}{\mu}$. Evaluating the derivatives, $Q'_\mu(0) = 1$ and $Q'_\mu(1 - \frac{1}{\mu}) = -\mu + 2 \in (-1, 1)$. Hence $x = 0$ is now a repelling fixed point and $x = 1 - \frac{1}{\mu}$ is an attractive fixed point.

Case 4 : $\mu = 3$

The fixed point $x = 1 - 1/3$ is still attractive [*exercise*].

Case 5 : $\mu > 3$

Both fixed points $x = 0$ and $x = 1 - \frac{1}{\mu}$ are now repelling.

Now, to find period 2 points, we need to solve $Q_\mu(Q_\mu(x)) = x$

$$\begin{aligned}
&\mu Q_\mu(x)(1 - Q_\mu(x)) = x \\
\implies &\mu^2 x(1-x)(1 - \mu x(1-x)) = x \\
\implies &-\mu^3 x^4 + 2\mu^3 x^3 + (-\mu^3 - \mu^2)x^2 + \mu^2 x = x \\
\implies &-\mu^3 x^4 + 2\mu^3 x^3 + (-\mu^3 - \mu^2)x^2 + (\mu^2 - 1)x = 0 \\
\implies &\underbrace{-x}_{\text{fixed point at } 0} \underbrace{(\mu x + 1 - \mu)}_{\text{fixed point at } 1 - \frac{1}{\mu}} (\mu^2 x^2 - (\mu^2 - \mu)x + (\mu + 1)) = 0
\end{aligned}$$

The period 2 points can be obtained via the quadratic equation

$$\begin{aligned}
x &= \frac{\mu^2 + \mu \pm \sqrt{(\mu^2 + \mu)^2 - 4\mu^2(\mu + 1)}}{2\mu^2} \\
&= \frac{\mu + 1 \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu}
\end{aligned}$$

We need the discriminant $\mu^2 - 2\mu - 3 = (\mu - 3)(\mu + 1) \geq 0$, so to have a period 2 point, we need $\mu > 3$. After much algebra, we get that $3 < \mu < 1 + \sqrt{6}$

1.2.3 The family $B_\mu(x) = \mu x - \lfloor \mu x \rfloor$ on $0 \leq \mu \leq 2$

The function has a fixed point at $x = 0$ which is attractive when $\mu < 1$ and repelling when $\mu > 1$ since μ represents the slope parameter of the line. At $\mu = 1$, every $x \in [0, 1)$ is a fixed point.

See `https://www.desmos.com/calculator/x2dimurcgm`

1.2.4 The family of tent functions

Tent functions, named after their shape, are defined as :

$$T_\mu(x) = \begin{cases} 2\mu x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2\mu x & \frac{1}{2} < x < 1 \end{cases}$$

Under iteration, these functions begin to show fractal-like behavior. See `https://www.desmos.com/calculator/pwj7grlq9a`

1.3 Bifurcations

Definition 1.9 (Bifurcation point)**.** *Let $f_\mu(x)$ be a family of functions. If the number of, or nature of (attractive/repelling) fixed and/or periodic points changes at $\mu = \mu*$, then we call $\mu*$ a bifurcation point.*

$\boxed{g_\mu(x) = x^2 + \mu \text{ has a bifurcation point at } \mu* = \frac{1}{4} \text{ as seen earlier - it goes from 2 fixed points to 0 fixed points.}}$ $\boxed{\#10}$

Definition 1.10 (Period doubling/pitchfork point)**.** *A bifurcation point that goes from an attractive fixed point to an attractive period 2 point or an attractive period n point to an attractive period $2n$ point is called a period doubling bifurcation point, sometimes called a pitchfork bifurcation point.*

Definition 1.11 (Tangent bifurcation point)**.** *We say a bifurcation point going from one attractive and one repelling fixed or periodic point to none is a tangent bifurcation point.*

1.4 Relations between periodic points

For a continous function over $\mathbb{R} \to \mathbb{R}$, there is a precise relationship between points of period n and period m.

Definition 1.12 (Sharkovsky order)**.** *Let $a \neg b$ mean a is before b in this order. Then the Sharkovsky Order is* $3 \neg 5 \neg 7 \neg 9 \neg ... \neg 2 \cdot 3 \neg 2 \cdot 5 \neg ... \neg 2^2 \cdot 3 \neg 2^3 \cdot 5 \neg ... \neg 2^k \cdot 3 \neg 2^k \cdot 5 \neg ... \neg 2^3 \neg 2^2 \neg 2^1 \neg 1$.

Equivalently, $2^{k_1} p_1 \neg 2^{k_2} p_2$ where p_1, p_2 are odd if :

1) $k_1 = k_2, 3 \leq p_1 < p_2$
2) $k_1 < k_2, 3 \leq p1, p_2$
3) $p_1 \neq 1, p_2 = 1$
4) $p_1 = p_2 = 1, k_1 > k_2$

Theorem 1.6 (Sharkovsky, 1964)**.** *Let $f(x)$ be a continuous function $\mathbb{R} \to \mathbb{R}$. If $f(x)$ has a periodic point of period n, then it will have a periodic point of period m, for all $n \neg m$. Moreover, this result is sharp - i.e., for any n, there exists a function f such that $f(x)$ has a periodic point of period n and $n \neg m$, but has no period point of period k for $k \neg m$.*

As Sharkovsky's theorem was not known to the west for many years, Li-Yorke proved the case for points of period 3 independently, with a much simpler proof.

Theorem 1.7 (Li-Yorke, 1975)**.** *Let $f(x)$ be continuous, $\mathbb{R} \to \mathbb{R}$. If $f(x)$ has a periodic point of period 3, then it has periodic points of all orders.*

To prove this, we recall some theorems from Calculus.

Lemma 1.8 (Extreme Value Theorem)**.** *Every continuous function $f(x)$ on a closed interval [a, b] achieves its maximum and minimum.*

This lemma says that there exists c, d such that $c, d \in [a, b]$ and $f(c) = \max\limits_{[a,b]} f(x)$ and $f(d) = \min\limits_{[a,b]} f(x)$.

Lemma 1.9 (Intermediate Value Theorem)**.** *Let $f(x)$ be a continous function on $[a, b]$. Then for any $c \in [f(a), f(b)]$, there exists $d \in [a, b]$ such that $f(d) = c$.*

Next, we generalize these lemmas from points to interval, which requires a definition for functions of intervals. Let $J = [a, b]$. Then $f(J) = \{f(x) \mid x \in J\}$.

Lemma 1.10. *Let $J = [a, b]$ be an interval and $f(x)$ be a continuous function. Then $f(J)$ is an interval.*

Proof. Pick c such that $c \in [a, b]$ and $f(c) = \max\limits_{[a,b]} f(x)$. Pick d similarly for the minimum, which we know exists by lemma 1.8.

For any $y \in [f(d), f(c)]$, there will exist an $e \in [c, d] \subseteq [a, b]$ such that $f(e) = y$ by lemma 1.9. Hence the entirety of $[f(d), f(c)] \in f(J)$, but there are no points outside $[f(d), f(c)]$ as $f(d)$ is the min and $f(c)$ is the max.

$\therefore f(J) = [f(d), f(c)]$ is an interval.

$\square$

Lemma 1.11. *Let $f(x)$ be continuous on J and $L \subseteq f(J)$. Then there exists a $J_0 \subseteq J$ such that $L = f(J_0)$.*

Proof. Let $L = [c, d]$ and $X = x_1, x_2, ..., x_n$ such that $f(x_i) = c$. Pick $Y = y_1, y_2, ...y_m$ similarly such that $f(y_j) = d$.

Order the set $X \cup Y$ and pick x_i, y_j that are adjacent in this order. The claim is that $J_0 = [x_i, y_j]$ work. Clearly, $L \subseteq f(J_0)$. Furthermore, if there exists $y \in f(J_0) \backslash L$, then either $f(y) > c$ or $f(y) > d$ and we have another crossing of the line $y = c$ or $y = d$.

$\square$

Lemma 1.12. *Let $f(x)$ be continuous on $J = [a, b]$ and $J \subseteq f(J)$. Then $f(x)$ has a fixed point in J.*

Proof. Consider $c \in J$ such that $f(c) = \max\limits_{[a,b]} f(x)$ and $d \in J$ such that $f(d) = \min\limits_{[a,b]} f(x)$. We know $f(c) \geq b \geq c$ and $f(d) \leq a \leq d$. If $f(c) = c$ or $f(d) = d$, then we are done.

Otherwise, consider $g(x) = f(x) - x$. We know that :

$$g(c) = f(c) - c > 0$$

$$g(d) = f(d) - d < 0$$

Hence, by the intermediate value theorem, there exists an $e \in [c, d] \subseteq [a, b]$ such that $g(e) = 0$. That is, $f(e) - e = 0 \implies f(e) = e$, which gives a fixed point.

$\square$

With these tools, we can now make our first statement about the implication of period 3 points on the existence of fixed points and period 2 points.

Theorem 1.13. *Let $f(x)$ be continuous on J, $f(a) = b, f(b) = c, f(c) = a$ where $a, b, c \in J$. Then $f(x)$ has a fixed point and a period 2 point.*

Proof. Without loss of generality, let $a < b < c$. Notice $\{a, b, c\} \subseteq f([a, c])$ so $[a, c] \subseteq f([a, c])$ by 1.9. By 1.12, the interval $[a, c]$ must contain a fixed point.

To find the period 2 point, notice that $f([b, c]) \supseteq [a, c] \supseteq [a, b]$. Then there will exist a smaller interval $[b_0, c_0] \subseteq [b, c]$ such that $f([b_0, c_0]) = [a, b]$. Furthermore, $f^{[2]}([b_0, c_0]) = f([a, b]) \supseteq [b, c] \supseteq [b_0, c_0]$. By 1.12, $f^{[2]}(x)$ has a fixed point in $[b_0, c_0]$, say e.

This fixed point is such that $f(e) \in [a, b]$ and $f^{[2]}(e) = e \in [b_0, c_0]$. The only way these overlap is if $b_0 = b$ and $e = b$. However, $f^{[2]}(b) = a \neq b$, so e cannot be a fixed point of $f(x)$. Therefore, it must be a period-2 point.

□

With this theorem, we now have sufficient tools to prove Li-Yorke (1.7).

Proof. Given the theorem above, it suffices to show that $f(x)$ has periodic points of period n, $n \geq 4$. The goal is to construct a point e such that :

1) $e, f(e), ..., f^{[n-2]}(e) \in [b, c]$
2) $f^{[n-1]}(e) \in (a, b)$
3) $f^{[n]}(e) = e$.

Notice $f([b, c]) \supseteq [a, c] \supseteq [b, c]$. By lemma 1.11 we know we can construct $[b_0, c_0] \subseteq [b, c]$ such that $f([b_0, c_0]) = [b, c]$. Next, $f^{[2]}([b_0, c_0]) = f([b, c]) \supseteq [a, c] \supseteq [b_0, c_0]$. Then, construct $[b_1, c_1] \subseteq [b_0, c_0]$ such that $f^{[2]}([b_1, c_1]) = [b_0, c_0]$. Similarly, we can construct $[b_2, c_2] \subseteq [b_1, c_1]$ such that $f^{[3]}([b_2, c_2]) = [b_1, c_1]$.

Repeat this process until we get $f^{[n-2]}([b_{n-3}, c_{n-3}]) = [b_{n-4}, c_{n-4}]$. So we have a set of nested intervals $[b_0, b_1] \supseteq [b_1, b_2] \supseteq ... \supseteq [b_{n-3}, c_{n-3}]$.

Note that $f^{[n-1]}([b_{n-3}, c_{n-3}]) \supseteq [a, b]$. Construct $[b_{n-2}, c_{n-2}]$ such that $f^{[n-1]}([b_{n-2}, c_{n-2}]) \supseteq [a, c] \supseteq [b_{n-2}, c_{n-2}]$.

Hence, there is a fixed point e. This is not a fake fixed point since $e, f(e), ..., f^{[n-2]}(e) > b$ and $f^{[n-1]}(e) < b$.

□

2 Chaos

2.1 Sensitivity, transitivity and density

Definition 2.1 (Sensitive dependence on initial conditions)**.** *A function $f(x)$ has sensitive dependence on initial conditions (abbreviated s.d.i.c.) at x_0 if there exists an $\epsilon > 0$ such that $\forall\ \delta > 0$, there exists a $y, |x_0 - y| < \delta$ and an n such that $|f^{[n]}(y) - f^{[n]}(x_0)| > \epsilon$.*

$y = x^2$ has s.d.i.c. at $x = 1$.

In this case, any ϵ will work, say 2. For any $\delta > 0$, pick $y = 1 + \frac{\delta}{2} > 1$. Clearly, through repeated exponentiation, there will be an n such that $f^{[n]}(y) > 3$. Then $|f^{[n]}(y) - f^{[n]}(1)| = |f^{[n]}(y) - 1| > 2 = \epsilon$, as required.

On the other hand, $y = x^2$ does not have s.d.i.c. at $x = 0$. #11

Assume that it did. Then an ϵ would exist to satisfy the definition for all δ. Now consider $\delta = \min(\frac{\epsilon}{2}, \frac{1}{2})$. Notice, for all y, that $|y| < \delta < 1$. We have $|f^{[n]}(y) - f^{[n]}(0)| = |f^{[n]}(y)| \leq |y| < \delta < \epsilon$. Hence, as this is true $\forall n, y$, we don't have s.d.i.c.

Definition 2.2 (S.D.I.C on intervals)**.** *A function $f(x)$ has s.d.i.c. on J if it has s.d.i.c. for all $x \in J$.*

Let $B(x) = \begin{cases} 2x & x < \frac{1}{2} \\ 2x - 1 & x \geq \frac{1}{2} \end{cases}$

We claim that $B(x)$ has s.d.i.c.

Proof. Case 1 : Assume $x = \frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $\frac{p}{2^n} \in [0, 1]$. Notice that $B(x)$ is of the form $\frac{p'}{2^{n-1}}$ for some p'.

Under iterations, $B^{[k]}(x) \to 0$ or 1 since at some point, $B^{[k]}(x)$ will be of the form $\frac{p''}{2^0}$. However this number is in the range $[0, 1]$ and also an integer, so it must be either 0 or 1. Furthermore, it can be 1 iff $x = 1$, so we ignore this case.

Notice that if x is not of that form, then $B^{[k]}(x) \not\to 0$ or 1.

Take $\epsilon = \frac{1}{2}$. For all $\delta > 0$ we can find y that is not of the form $\frac{p}{2^n}$ and $|x - y| < \delta$. Then $B^{[n]}(x) = 0$ but $B^{[n]}(y) \neq 0$. If $B^{[n]}(y) > \frac{1}{2}$ then we are done. Otherwise, the next iterate $B^{[n+1]}(y) = 2B^{[n]}(y)$. Keep iterating k times until $B^{[n+k]}(y) > \frac{1}{2}$ and $B^{[n+k]}(x)$ will still be 0. #12

$\therefore |B^{[n+k]}(y) - B^{[n+k]}(x)| > |\frac{1}{2} - 0| > \epsilon$ so x has s.d.i.c.

Case 2 : Assume x is not of the form $\frac{p}{2^n}$. Again, take $\epsilon = \frac{1}{2}$. For all $\delta > 0$ we can find y that is of the form $\frac{p}{2^n}$ and $|x - y| < \delta$. Thus, we can repeat the same argument as in the previous case, switching x instead of y.

Since every x has s.d.i.c, so does $B(x)$. □

Definition 2.3 (Lyapunov exponent). *We define the Lyapunov exponent of $f(x)$ at x as*

$$\lambda(x) = \lim_{n\to\infty} \frac{\ln|(f^{[n]})'(x)|}{n}$$

The larger the Lyapnuov exponent is, the more sensitive the function is at x, hence the usefulness of this measure in studying chaotic systems. Furthermore, the more negative this number is, the more attractive a fixed period point.

Let $B(x)$ be defined as in the previous example and x not be of the form $\frac{p}{2^n}$. Then $B(x)$ is differentiable at x and $|B'(x)| = 2, |B^{[2]'}(x)| = 4, |B^{[n]'}(x)| = 2^n$.

Thus,

$$\lambda(x) = \lim_{n\to\infty} \frac{|\ln((f^{[n]'})(x))|}{n} = \lim_{n\to\infty} \frac{|\ln(2^n)|}{n} = \ln(2) > 0$$

#13

Definition 2.4 (Chaotic). *We say a function f is chaotic on J if either*

1) f has s.d.i.c. for all $x \in J$ or
2) f has $\lambda(x) > 0$ for all $x \in J$

Definition 2.5 (Transitivity). *We say $f(x)$ is transitive on J if for all open intervals $U, V \in J$, there exists an n such that $f^{[n]}(u) \cap V \neq \emptyset$ where $u \in U$.*

$B(x)$ is transitive on [0, 1]. In fact, we can show a stronger statement that for any U, there exists n such that $B^{[n]}(U) = [0, 1)$.

Proof. Any interval U will contain a point of the form $\frac{p}{2^n}$. We've shown that under iteration, $B^{[n]}(\frac{p}{2^n}) = 0$. Hence, $0 \in B^{[n]}(U)$ and there will be an interval $[0, u) \subseteq B^{[n]}(U), u > 0$.

In the next iteration, we will have that $B^{[n+1]}(U) \supseteq [0, 2u)$ and for k more iterations, $B^{[n+k]}(U) \supseteq [0, 2^k u)$. When $2^k u > 1$, we will have $B^{[n+k]}(U) \supseteq [0, 1)$. □

#14

Definition 2.6 (Dense). *We say $A \subseteq J$ is dense in J if for all open intervals $U \in J$, we have $U \cap A \neq \emptyset$.*

$\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R}$ is dense in $\mathbb{R}$.
However, $\mathbb{N}$ is not dense in $\mathbb{R}$ (take for example $(\frac{1}{2}, \frac{2}{3})$).

#15

Theorem 2.1. *Let $f : J \to J, J$ a bounded closed interval. Then $f(x)$ is transitive on J iff there exists a point x_0 such that $\{f^{[n]}(x_0)\}_{n=1}^{\infty}$, the orbit, is dense in J.*

We've seen that $B(x)$ is transitive on $[0, 1]$ so we know that there exists a x_0 such that the orbit of $B^{[n]}(x_0)$ will be dense on $[0, 1]$. This x_0 will not be rational since every $\mathbb{Q}$ is eventually fixed or periodic, which implies a finite orbit and finite orbits cannot possibly be dense.

#16

Proof. ($\Leftarrow$) Assume that there exists a point x_0 such that $\{f^{[n]}(x_0)\}_{n=1}^{\infty}$ is dense in J. Let $U, V \subseteq J$ be open intervals. As $\{f^{[n]}(x_0)\}_{n=1}^{\infty}$ is dense, there will exist an n_0 such that $f^{[n_0]}(x_0) \in U$. (In fact, there will exist an infinite number of such n_0).

Similarly, there will exist an infinite number of m_0 such that $f^{[m_0]}(x_0) \in V$. We can assume without loss of generality that $n_0 < m_0$ since there are infinitely many of each.

So we have that

$$\begin{aligned} & f^{[n_0]}(x_0) \in U \text{ and } f^{[m_0]}(x_0) \in V \\ \implies & f^{[m_0-n_0]}(f^{[n_0]}(x_0)) \in f^{[m_0-n_0]}(U) \\ \implies & f^{[m_0]}(x_0) \in f^{[m_0-n_0]}(U) \cap V \\ \implies & f^{[m_0-n_0]}(U) \cap V \neq \emptyset \end{aligned}$$

$\therefore f(x)$ is transitive.

□

Proof. ($\Rightarrow$) I lost my notes for this part, so the proof is ommitted. *sorry!*

□

Definition 2.7 (Strong Chaos)**.** *A function f on a finite interval J is strongly chaotic if :*

1) f is chaotic
2) f has a dense set of periodic points
3) f is transitive

2.2 Conjugacy

Often, we are interested in analysing the properties of a function, say whether it is transitive, but the task is too difficult. It would be useful to have a tool that allows us to simplify the problem.

Definition 2.8 (Homeomorphism)**.** *Let J and K be intervals. The function $f : J \to K$ is a homeomorphism from J to K if f is one-to-one, onto and both f and f^{-1} are continuous.*

Note that if f is a homeomorphism from J to K, then f^{-1} must also be an homeomorphism from K to J - this property is symmetric.

Definition 2.9 (Conjugacy)**.** *Let J, K be intervals and suppose $f : J \to J$ and $g : K \to K$. Then f and g are conjugate if there exists a homeomorphism $h : J \to K$ such that $h \circ f = g \circ h$, which we write as $f \approx_h g$.*

Because of the symmetry of homeomorphism, when $f \approx_h g$, $g \approx_{h^{-1}} f$.

Theorem 2.2 (Properties of conjugates)**.** *If $f \approx_n g$, then :*

1) $h \circ f^{[n]} = g^{[n]} \circ h$
2) x is a fixed/period-n point of f iff $h(x)$ is a fixed/period-n point for g
3) x has dense orbit for f iff $h(x)$ has dense orbit for g
4) f is transitive iff g is transitive
5) A fixed/periodic point x is attractive for f iff $h(x)$ is also for g.

Proof. (1) Proof by induction. In the base case, we already know that $h \circ f = g \circ h$. Assume $h \circ f^{[n-1]} = g^{[n-1]} \circ h$.

$$\begin{aligned}
& h \circ f^{[n]} \\
&= (h \circ f) \circ f^{[n-1]} \\
&= (g \circ h) \circ f^{[n-1]} \\
&= g \circ (h \circ f^{[n-1]}) \\
&= g \circ (g^{[n-1]} \circ h) \\
&= g^{[n]} \circ h
\end{aligned}$$

□

Proof. (2) Assume p is a period-n point of f, such that $f^{[n]}(p) = p$. Using the previous property, we get that :

$$h(f^{[n]}(p)) = g^{[n]}(h(p))$$

and

$$h(f^{[n]}(p)) = h(p)$$

$\therefore g^{[n]}(h(p)) = h(p)$ so $h(p)$ is a period-n point of g.

□

Proof. (3) Assume x has dense orbit. Let $\left\{f^{[n]}(x)\right\}_{n=0}^{\infty}$ be the orbit and consider

$$\left\{g^{[n]}(h(x))\right\}_{n=0}^{\infty} = \left\{h \circ f^{[n]}(x)\right\}_{n=0}^{\infty} = h\left(\left\{f^{[n]}\right\}_{n=0}^{\infty}\right)$$

Let I_2 be an open interval in K (note $g : K \to K$). Given that h is continuous, consider $h^{-1}(I_2)$ as h^{-1} is continuous, one-to-one and onto, $h^{-1}(I_2)$ is an open interval in J, hence $h^{-1}(I_2) \cap \{f^{[n]}(x)\} \neq \emptyset$. Applying h gives $h(h^{-1}(I_2)) \cap h(\{f^{[n]}(x)\}) \neq \emptyset \implies I_2 \cap \left\{g^{[n]}(h(x))\right\}_{n=0}^{\infty} \neq \emptyset$.

□

Proof. (4) Immediate from (3) by theorem 2.1.

□

Proof. (5) Let p be an attractive fixed point. This means there exists I, $p \in I$ such that for all $x \in I$, $f^{[n]}(x) \to p$ as $n \to \infty$. Thus, we see that $h(f^{[n]}(x)) \to h(p)$. Equivalently, $g^{[n]}(h(x)) \to h(p)$ for all $h(x) \in h(I)$ and the interval $h(I)$ contains $h(p)$, so $h(p)$ is attractive for g.

□

The functions $g_v(x) = x^2 + v$, $Q_\mu(x) = \mu x(1-x)$ are conjugate for $v = \frac{2\mu - \mu^2}{4}$ by a linear homeomorphism. That is, $h \circ Q_\mu(x) = g_v(h(x))$ where $h(x) = ax + b$. Find $h(x)$.

#17

$$\begin{aligned} & h \circ Q_\mu(x) \\ &= h(\mu x(1-x)) \\ &= a\mu x(1-x) + b \\ &= -a\mu x^2 + a\mu x + b \end{aligned} \qquad \begin{aligned} & g_v(h(x)) \\ &= g_v(ax+b) \\ &= (ax+b)^2 + v \\ &= a^2x^2 + 2abx + b^2 + \frac{2\mu - \mu^2}{4} \end{aligned}$$

By matching coefficients, we see that $a = -\mu$ (for x^2), $a\mu = 2ab \implies b = \frac{\mu}{2} \implies h(x) = -\mu x + \frac{\mu}{2}$.

Let $Q_4(x) = 4x(1-x)$ and $T(x) = \begin{cases} 2x & x < \frac{1}{2} \\ 2 - 2x & x \geq \frac{1}{2} \end{cases}$

We know from previous examples that $T(x)$ has periodic points of all orders, none of which are attractive and that $T(x)$ is transitive. These properties are harder to prove directly for $Q_4(x)$ - unless we can find a homeomorphism between $T(x)$ and $Q_4(x)$.

We claim that $h(x) = \sin^2(\frac{\pi}{2}x)$ as $h : [0,1] \to [0,1]$ is such a homeomorphism.

#18

$$\begin{aligned} h \circ T(x) &= h\left(\begin{cases} 2x & x < \frac{1}{2} \\ 2 - 2x & x \geq \frac{1}{2} \end{cases}\right) \\ &= \begin{cases} \sin^2(\frac{\pi}{2} \cdot 2x) & x < \frac{1}{2} \\ \sin^2(\frac{\pi}{2} \cdot (2x - 2)) & x \geq \frac{1}{2} \end{cases} \\ &= \begin{cases} \sin^2(\pi x) & x < \frac{1}{2} \\ \sin^2(-\pi x) & x \geq \frac{1}{2} \end{cases} \\ &= \sin^2(\pi x) \end{aligned}$$

$$\begin{aligned} Q_4(h(x)) &= 4\sin^2(\frac{\pi}{2}x)(1 - \sin^2(\frac{\pi}{2}x)) \\ &= 4\sin^2(\frac{\pi}{2}x)(\cos^2(\frac{\pi}{2}x)) \\ &= (2\sin^2(\frac{\pi}{2}x)(\cos^2(\frac{\pi}{2}x)))^2 \\ &= \sin^2(\pi x) \end{aligned}$$

3 Fractals

3.1 The Cantor Set

After having covered the basics of iterated functions, chaos and a number of tools along the way, we are finally ready to begin the study of our first fractal : the Cantor Set. There are many ways to describe the Cantor Set, with various degrees of formality. To tie it to the material covered so far, we will use the help of a variant of an iterated function we've seen before.

Consider $T_\mu(x) = \begin{cases} 2\mu x & x < \frac{1}{2} \\ 2\mu(1-x) & x \geq \frac{1}{2} \end{cases}$

We already know that for $\mu < 1$, this looks like $Q_4(x)$ (using conjugates) which we understand. What about $\mu > 1$?

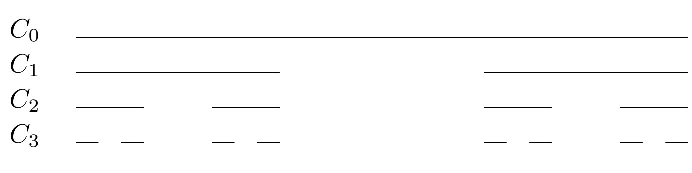
Consider $\mu = \frac{3}{2}$ such that $T_{\frac{3}{2}}(x) = \begin{cases} 3x & x < \frac{1}{2} \\ 3 - 3x & x \geq \frac{1}{2} \end{cases}$ and we now take $T_{\frac{3}{2}} : \mathbb{R} \to \mathbb{R}$.

We do the iterates look like?

$$\left\{ {T_{\frac{3}{2}}}^{[n]}\left(\frac{1}{5}\right) \right\}_{n=0}^{\infty} = \left\{ \frac{1}{5}, \frac{3}{5}, \frac{6}{5}, -\frac{3}{5}, -\frac{9}{5}, -\frac{27}{5}, ... \right\}$$
$$\left\{ {T_{\frac{3}{2}}}^{[n]}\left(\frac{1}{4}\right) \right\}_{n=0}^{\infty} = \left\{ \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, ... \right\}$$
$$\left\{ {T_{\frac{3}{2}}}^{[n]}\left(\frac{1}{3}\right) \right\}_{n=0}^{\infty} = \left\{ \frac{1}{3}, 1, 0, 0, 0, ... \right\}$$
$$\left\{ {T_{\frac{3}{2}}}^{[n]}\left(\frac{1}{2}\right) \right\}_{n=0}^{\infty} = \left\{ \frac{1}{2}, \frac{3}{2}, -\frac{9}{2}, -\frac{-27}{2}, ... \right\}$$

Some points are fixed, some points are eventually fixed and we claim that most $x \in [0,1]$ go to $-\infty$. In fact, it will if an iterate is ever < 0. Furthermore, if ${T_{\frac{3}{2}}}^{[n]}(x) > 1$ for some n then ${T_{\frac{3}{2}}}^{[n+1]}(x) < 0$ and ${T_{\frac{3}{2}}}^{[k]}(x) \to -\infty$.

Therefore, we are interested in the long-term behavior of each point. Define $C_n = \{\ x \mid {T_{\frac{3}{2}}}^{[n]}(x) \in [0,1]\ \}$ and $C = \{\ x \mid {T_{\frac{3}{2}}}^{[n]}(x) \not\to -\infty\ \}$. Then the progression of C_n looks like the picture below:



Notice that at each step, $C_n =$ intervals in C_{n-1} with the middle third removed and $C = \bigcap_{n=0}^{\infty} C_n$. By this definition, any endpoint of any interval of any C_n is in the cantor set - i.e., $0, \frac{1}{3}, \frac{1}{9}, \frac{2}{3}, \frac{7}{9}...$

Any eventually fixed or periodic point of $T_{\frac{3}{2}}$ is in the Cantor Set, all of which are rational.

There are many other properties of the Cantor Set which we can study.

Theorem 3.1 (The Cantor Set)**.** *Let C be the Cantor Set.*

1) C is totally disconnected.
2) Every point in C is a limit point of points in C.
3) C has no area, or content 0.
4) C has dimension $\frac{\log 2}{\log 3} < 1$.
5) C is the set of numbers $0.a_1a_2a_3...$ in base 3 with $a_i \in 0, 2$.
6) C has an uncountable number of points.
7) C contains an irrational number.
8) $T_{\frac{3}{2}}(x)$ has s.d.i.c. on C.
9) $T_{\frac{3}{2}}(x)$ has a point of dense orbit in C.

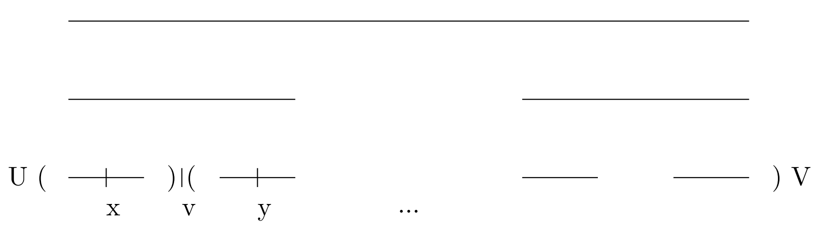
Definition 3.1 (Totally disconnected)**.** *We say a set A is totally disconnected if for all $x, y \in A, x \neq y$, there exists an open interval U and V such that :*

1) $x \in U, y \in V$
2) $A \subseteq U \cup V$
3) $U \cap V = \emptyset$

Theorem 3.2. *C is totally disconnected.*

Proof. There exists an n such that x, y are in different intervals in C_n. To ensure this result, pick n such that $\frac{1}{3^n} < |x - y|$. Pick v that is not in C_n but is between these two intervals. Then $v \notin C$ also.

Finally, take $U = (-1, v)$ and $V = (v, 2)$ and notice that $x \in U$ and $y \in V$, $C \subseteq U \cap V$ and $U \cap V = \emptyset$. This satisfies the definition.



□

Theorem 3.3. *Every point in C is a limit point of points in C.*

In other words, $\forall x \in C$ there exists $\{x_n\}_{n=0}^{\infty}, x_n \in C, x_n \neq x$ and $\lim_{n\to\infty} x_n = x$.

Proof. Assume that x is not the left endpoint of any interval of any C_n. Pick x_n to be the leftmost endpoint of the interval in which x belongs. From this we observe that :

1) $x_n \in C$, since endpoints are in C.
2) $|x_n - x| < \frac{1}{3^n}$ so $\lim_{n\to\infty} x_n = x$
3) $x_n \neq x$ for all n as x is not a left endpoint.

If x was the left endpoint, take the x_n to be the right endpoint. □

Definition 3.2 (Content 0)**.** *We say a set A has content 0 if for all $\epsilon > 0$, there exists finitely many $I_k = [a_k, b_k]$ such that 1) $A \subseteq \bigcup_{k=0}^{N} I_k$ and 2) $\sum_{k=0}^{N} |I_k| = \sum_{k=0}^{N}(b_k - a_k) < \epsilon$.*

Let $A = \{x_1, x_2, ..., x_n\}$, a set with a finite number of points. Take $I_k = [x_k - \frac{\epsilon}{3n}, x_k + \frac{\epsilon}{3n}]$. So $A \subseteq \bigcup I_k$ and $\sum I_k = \frac{2}{3}\epsilon < \epsilon$. Thus, finite sets of points have content 0. #19

Theorem 3.4. *C has content 0.*

Proof. Notice C_n is a collection of a finite number of intervals that cover C. Each C_n will have 2^n intervals of length $\frac{1}{3^n}$. Hence $\sum_{I\in C_n} = (\frac{2}{3})^n$. Then clearly, for any $\epsilon > 0$, we can find an n such that $(\frac{2}{3})^n < \epsilon$, and we just used C_n to cover C. Hence C has content 0.

□

Theorem 3.5. $C = \{0.\underbrace{0.a_1a_2a_3...}_{\textit{base 3}} \mid a_i \in 0, 2\}$

Definition 3.3 (Countable)**.** *A set S is countable if there exists an onto map $f : \mathbb{N} \to S$.*

That is, we are looking for a map f such $\{f(n)\}_{n=0}^{\infty} = S$. Note that this map doesn't need to be one-to-one, so finite sets are countable. An alternatve and equivalent way of saying this is that we can find an order for a sequence $s_0, s_1, s_2, ...$ such that $S = \{s_n\}_{n=0}^{\infty}$. W

The set $\mathbb{N}$ is trivially countable using $f(n) = n$.

The integers $\mathbb{Z}$ is countable. Consider the sequence 0, 1, -1, 2, -2, ...

The set $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ is countable. See `http://personal.maths.surrey.ac.uk/st/H.Bruin/MMath/Cardinality.html`

The rationals $\mathbb{Q}$ is countable. We know $\mathbb{Z} \times \mathbb{Z}$ is countable so $\exists\ f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ - i.e., $f(n) = (a_n, b_n)$. #20

Use $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ with $g((a_n, b_n)) = \begin{cases} \frac{a_n}{b_n} & b_n \neq 0 \\ 42 & b_n = 0 \end{cases}$.

This gives $g \circ f : \mathbb{N} \to \mathbb{Q}$ is an onto map.

$\therefore \mathbb{Q}$ is countable.

Definition 3.4 (Uncountable)**.** *A set S is uncountable is S is not countable.*

$\boxed{\text{(exercise) Let } A \subseteq B\text{. If } B \text{ is countable so is } A\text{. If } A \text{ is uncountable so is } B\text{.}}$ $\boxed{\#21}$

Theorem 3.6. *The Cantor Set is uncountable.*

Proof. We will assume C is countable and derive a contradiction.

Assume there exists an onto map $f : \mathbb{N} \to C$:

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}... & a_{1i} &\in 0, 2 \\ f(2) &= 0.a_{21}a_{22}a_{23}... & a_{2i} &\in 0, 2 \\ &\vdots \\ f(n) &= 0.a_{n1}a_{n2}a_{n3}... & a_{ni} &\in 0, 2 \end{aligned}$$

The goal is to construct a new number that is not any of the above. Consider $y = 0.b_1b_2b_3...$ where $b_n = \begin{cases} 0 & a_{nn} = 2 \\ 2 & a_{nn} = 0 \end{cases}$

Notice that for all n, $f(n) \neq y$ as they differ in the nth digit. Furthermore, since it contains only the digits 0, 2, then by theorem 3.5, $y \in C$.

$\therefore$ there does not exist an onto map $f : \mathbb{N} \to C \implies C$ is uncountable. $\square$

As corollary is that since $C \subseteq \mathbb{R}$, then $\mathbb{R}$ is uncountable.

Theorem 3.7. *C contains an irrational number.*

Proof. This can be proved in two ways. The first is to notice that $\mathbb{Q}$ is countable but C isn't, so $C \cap \mathbb{Q} \neq \emptyset$ and C will contain an irrational.

Alternatively, we can construct such an irrational number. Consider

$$x = 0.2\underbrace{\quad}_{0}2\underbrace{0}_{1}2\underbrace{00}_{2}2\underbrace{000}_{3}2\underbrace{0000}_{4}2...$$

All rational numbers eventual repeat themselves - i.e., $0.a_1a_2a_3...\overline{a_na_{n+1}...a_{n+k}}$. This point does not, hence it is irrational.

$\square$

Theorem 3.8. $T_{\frac{3}{2}}(x) = \begin{cases} 3x & x < \frac{1}{2} \\ 3 - 3x & x \geq \frac{1}{2} \end{cases}$ *has sensitive dependence on initial conditions on C.*

Proof. Take $\epsilon = \frac{1}{4}$. Notice that if we have two points $x, y \in C$ where $x < \frac{1}{2} < y$ or $y < \frac{1}{2} < x$, then $|x - y| \geq \frac{1}{3} > \frac{1}{4}$, so we can assume either $x, y < \frac{1}{2}$ or $x, y > \frac{1}{2}$. In that case, $|T(x) - T(y)| = 3|x - y|$.

For any $x \in C$, take $y \in C, y \neq x, |x-y| < \delta$. Notice that either $|T(x) - T(y)| = 3|x-y|$ or $x < \frac{1}{2} < y$ or $y < \frac{1}{2} < x$. So either $|T(x) - T(y)| > \frac{1}{4}$ or $|T^{[2]}(x) - T^{[2]}(y)| = 3^2|x-y|$. Continuing this iteration process, there will exist an n such that $|T^{[n]}(x) - T^{[n]}(y)| = 3^n|x-y| > \frac{1}{4}$. This proves the result.

$\square$

Theorem 3.9. $T_{\frac{3}{2}} = \begin{cases} 3x & x < \frac{1}{2} \\ 3 - 3x & x \geq \frac{1}{2} \end{cases}$ *is transitive on C/has a point of dense orbit in C.*

Proof. Consider $0.a_1a_2a_3... \in C$, written in base 3, where $a_i \in 0, 2$

Note that $1 = 0.222...$, so $T_{\frac{3}{2}}(0.a_1a_2a_3...) = \begin{cases} 0.a_2a_3a_4... & a_1 = 0 \\ 0.\overline{a}_2\overline{a}_3\overline{a}_4... & a_1 = 2 \end{cases}$ where $\overline{a}_i = \begin{cases} 0 & a_i = 2 \\ 2 & a_i = 0 \end{cases}$.

Now, consider $0.\underbrace{0 * 2}_{1} * \underbrace{00 * 02 * 20 * 22}_{2} *....$ Choose each $*$ as necessary to get an even number of flip at every $*$. That is, choose
the first $*$ such that $T^{[2]}(x) = 2 * 00 * 02 * 20 * 22...$
the second $*$ such that $T^{[4]}(x) = 00 * 02 * 20 * 22...$
the third $*$ such that $T^{[7]}(x) = 02 * 20 * 22...$

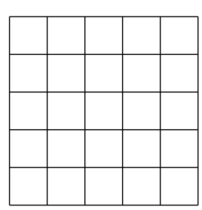
This point has a dense orbit since it contains every combination of 0 and 2 in its digits, so it gets arbitrarily close to every point in C, which proves the result. $\square$

Now, there is only one property of the Cantor Set which we did not yet prove : the dimension of this set. How can a set have a fractional dimension?

3.2 Box-Counting dimension a.k.a Capacity

Definition 3.5 (N_s)**.** *Let $S \subseteq \mathbb{R}^n$. Define $N_S(\epsilon)$ to be the minimal number boxes of size of $\epsilon \times \epsilon \times ... \times \epsilon$ boxes used to cover S.*

Let $S = [0,1] \times [0,1] \times 0 \subseteq \mathbb{R}^3 = \{ (x,y,0) \mid 0 \leq x, y \leq 1 \}$. Then $N_S(\frac{1}{k}) = k^2$. Notice the ^{2}, which is the dimension of a plane. #22



$k \times k$ grid

Let $S = \{(x,y) \mid x^2 + y^2 = 1 \} \subseteq \mathbb{R}^2$. If we place boxes evenly around the circle, we will need approximately $N_S(\epsilon) \approx (2\pi\frac{1}{\epsilon})^1$. Notice the ^{1} - while the shape is defined in a two-dimensional space, a circle is effectively a one-dimensional line. #23

Definition 3.6 (Box-counting dimension). *We define the capacity or box-counting dimension as*

$$\dim_C(S) = \lim_{\epsilon \to 0} \frac{\ln(N_S(\epsilon))}{\ln(\frac{1}{\epsilon})}$$

Again, take $S = [0,1] \times [0,1] \times 0 \subseteq \mathbb{R}^3$. For now, assume $\epsilon = \frac{1}{k}$ as $k \to \infty$, which we will justify later. Then

$$\lim_{k\to\infty} \frac{\ln(N_S(\frac{1}{k}))}{\ln(k)} = \lim_{k\to\infty} \frac{\ln k^2}{\ln k} = \lim_{k\to\infty} \frac{2\ln k}{\ln k} = 2$$

#24

Theorem 3.10. *Let $0 < r < 1$. If $\lim_{k\to\infty} \frac{\ln N_S(r^k)}{\ln(1/r^k)}$ exists, then the capacity dimension will exist and will equal this limit. Moreover, the converse is true - if $\dim_C(S)$ exist, then so does this limit.*

Proof. If $\dim_C(S)$ exists, then $\lim_{k\to\infty} \frac{\ln(N_S(r^k))}{\ln(1/r^k)} = \dim_C(S)$ for all r. This proves ($\Leftarrow$).

Assume $\exists r$ such that $\lim_{k\to\infty} \frac{\ln(N_S(r^k))}{\ln(1/r^k)}$ exists and equals L. Pick $\epsilon > 0$.

There will exist a k such that $r^{k+1} \le \epsilon \le r^k$. Notice $N_S(r^{k+1}) \ge N_S(\epsilon) \ge N_S(r^k)$. Hence $\ln N_S(r^{k+1}) \ge \ln N_S(\epsilon) \ge \ln N_S(r^k)$ as ln is an increasing function. Furthermore,

$$\begin{aligned} & r^{k+1} \le \epsilon \le r^k \\ \implies & \frac{1}{r^{k+1}} \ge \frac{1}{\epsilon} \ge \frac{1}{r^k} \\ \implies & \ln\frac{1}{r^{k+1}} \ge \ln\frac{1}{\epsilon} \ge \ln\frac{1}{r^k} \\ \implies & \frac{1}{\ln\frac{1}{r^{k+1}}} \le \frac{1}{\ln\frac{1}{\epsilon}} \le \frac{1}{\ln\frac{1}{r^k}} \end{aligned}$$

Combining these gives

$$\frac{\ln N_S(r^k)}{\ln\frac{1}{r^{k+1}}} \le \frac{\ln N_S(\epsilon)}{\ln\frac{1}{\epsilon}} \le \frac{\ln N_S(r^{k+1})}{\ln\frac{1}{r^k}}$$

Notice $\ln(a^k) = k\ln a = \frac{k}{k+1}(k+1)\ln(a) = \frac{k}{k+1}\ln a^{k+1}$. This gives

$$\frac{k}{k+1}\frac{\ln N_S(r^k)}{\ln\frac{1}{r^k}} \le \frac{\ln N_S(\epsilon)}{\ln\frac{1}{\epsilon}} \le \frac{k+1}{k}\frac{\ln N_S(r^{k+1})}{\ln\frac{1}{r^{k+1}}}$$

So in the limit, $1 \cdot L \le \frac{\ln N_S(\epsilon)}{\ln\frac{1}{\epsilon}} \le 1 \cdot L$ by the squeeze theorem, hence $\lim_{\epsilon\to\infty} \frac{\ln N_S(\epsilon)}{\ln\frac{1}{\epsilon}} = L$

□

Let C be the Cantor Set. Let $r = \frac{1}{3}$. Notice that $N_C((\frac{1}{3})^k) = 2^k$, since at every iteration C_n, every intervals gets split in two, so the number of intervals double. So

$$\lim_{k\to\infty} \frac{\ln N_C((\frac{1}{3})^k)}{\ln 3^k} = \lim_{k\to\infty} \frac{\ln 2^k}{\ln 3^k} = \lim_{k\to\infty} \frac{k \ln 2}{k \ln 3} = \frac{\ln 2}{\ln 3}$$

This proves our last property for Cantor Sets. #25

3.3 Numerically estimating dimension

Very often, it is not possible to derive the box-counting dimension of an object directly. It may be that obtaining a convenient mathematical representation is not possible, or one may not exist at all, such as with picture of physical shapes such as clouds, rocks, blood vessels, etc.

Method for numerical estimation of box-counting dimension

Plot the points $(\ln(\frac{1}{\epsilon}), \ln(N_S(\epsilon)))$ for various values of ϵ. Find a line that fits those points using methods such as least squares. The slope of this line is the dimension.

The reason this works is that if we have an object of dimension d, then $N_S(\epsilon) \approx c(\frac{1}{\epsilon})^d$, where c is some constant. So $(\ln(\frac{1}{\epsilon}, \ln(N_S(\epsilon))) \approx (\ln(\frac{1}{\epsilon}, \ln(c) + d\ln(\frac{1}{\epsilon}))$ and the slope is d, as required.

4 Multi-dimensional Fractals

4.1 Dynamics of Linear Functions

Consider a linear system $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(\vec{v}) = A\vec{v} + \vec{b}$, where A is an $n \times n$ matrix and $\vec{v}, \vec{b}$ are vectors in $\mathbb{R}^n$.

We would like to study properties of this system. When do we have fixed points? When are they attractive or repelling?

Let $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} x + 2y + 4 \\ 2x + y + 6 \end{bmatrix}$

This has a fixed point if $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$. So we need to solve the linear system

$$\begin{aligned} x + 2y + 4 &= x & \implies && 2y + 4 &= 0 & \implies && y &= -2 \\ 2x + y + 6 &= y & && 2x + 6 &= 0 & && x &= -3 \end{aligned}$$

So $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ is a fixed point.

#26

In general,

$$\begin{aligned} f(\vec{v}) = A\vec{v} + \vec{b} &= \vec{v} \\ \implies A\vec{v} - \vec{v} &= -\vec{b} \\ \implies (A - I)\vec{v} &= -\vec{b} \\ \implies \vec{v} &= -(A - I)^{-1}\vec{b} \quad \text{if } (A - I) \text{ is invertible.} \end{aligned}$$

Theorem 4.1. *If $A - I$ is invertible, then $A\vec{v} + \vec{b}$ has a fixed point.*

Now, are those fixed points attractive or repelling? To answer that question, we need to look at eigenvalues and eigenvectors.

To find the eigenvalues in the previous example, compute

$$\det\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{bmatrix}\right) = (1-\lambda)^2 - 4 = (\lambda-3)(\lambda+1)$$

Which gives eigenvalues 3, -1.

To find the eigenvectors, notice
$\det\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} - 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix} \implies$ eigenvector $\begin{bmatrix}1\\ 1\end{bmatrix}$ and
$\det\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} + 1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix} \implies$ eigenvector $\begin{bmatrix}1\\ -1\end{bmatrix}$

Note that the eigenvectors are linearly independent and span $\mathbb{R}^n$.

$$\begin{aligned}
\text{Let } \begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}-3\\ 2\end{bmatrix} + s\begin{bmatrix}1\\ 1\end{bmatrix} + t\begin{bmatrix}1\\ -1\end{bmatrix}\\
f\left(\begin{bmatrix}x\\ y\end{bmatrix}\right) &= \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\left(\begin{bmatrix}-3\\ 2\end{bmatrix} + s\begin{bmatrix}1\\ 1\end{bmatrix} + t\begin{bmatrix}1\\ -1\end{bmatrix}\right) + \begin{bmatrix}4\\ 6\end{bmatrix}\\
&= \underbrace{\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}-3\\ 2\end{bmatrix} + \begin{bmatrix}4\\ 6\end{bmatrix}}_{\text{fixed point}} + s\underbrace{\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}}_{\text{eigen}} + t\underbrace{\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}}_{\text{eigen}}\\
&= \begin{bmatrix}-3\\ -2\end{bmatrix} + 3s\begin{bmatrix}1\\ 1\end{bmatrix} - t\begin{bmatrix}1\\ -1\end{bmatrix}
\end{aligned}$$ #27

In general, $f^{[n]}\left(\begin{bmatrix}x\\ y\end{bmatrix}\right) = \begin{bmatrix}-3\\ -2\end{bmatrix} + 3^n s\begin{bmatrix}1\\ 1\end{bmatrix} + (-1)^n t\begin{bmatrix}1\\ -1\end{bmatrix}$.

If $s \neq 0$, then this point is repelled away from the fixed point $\begin{bmatrix}-3\\ -2\end{bmatrix}$.

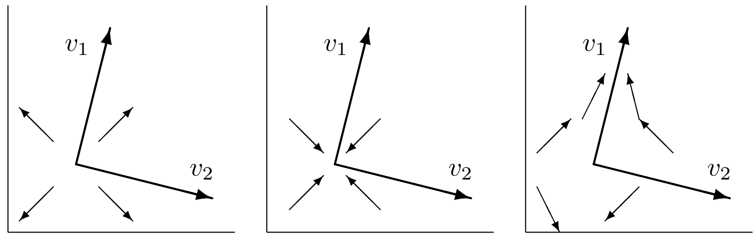
If $s = 0$, then this point will be periodic (assuming $t \neq 0$, which would then only be the fixed point) and have period 2.

From this example, we see that eigenvectors give us a useful "direction" in which the behavior of the function becomes clear.

Theorem 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(\vec{v}) = A\vec{v} = \vec{b}$. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A, real, and let $v_1, ..., v_n$ be the eigenvectors that span $\mathbb{R}_n$. Assume that the fixed point exists. Then*

1) If $|\lambda_i| > 1$ for all i, then the fixed point is repelling.
2) If $|\lambda_i| < 1$ for all i, then the fixed point is attractive.

On the other hand, if some eigenvalues > 1 and others < 1, interesting things can happen.



$\lambda_1, \lambda_2 > 1$ $\quad$ $0 < \lambda_1, \lambda_2 < 1$ $\quad$ $0 < \lambda_1 < 1 < \lambda_2$

Notice that in the last case, points are attracted along the line formed by v_1 but repelled by the lined formed by v_2.

Find a linear function with an attractive fixed point at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find an attractive fixed point, we need $f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we need $e = 1 - a - b$ and $f = 1 - c - d$. #28

There are many options - we can pick $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, which has easy eigenvalues.

So $f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4.2 Complex Eigenvalues

For now, assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ has a fixed point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. For example, $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$.

This has eigenvalues $\pm i$ and rotates $\begin{bmatrix} x \\ y \end{bmatrix}$ by $90°$ and $f^{[4]}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$.

This is also a counter example against a generalization of Sharkovsky's theorem in $\mathbb{R}^2$ since all points are period 4, but there is no point of period 2 except for the fixed point.

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ has a fixed point at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, eigenvalues $\pm\lambda i$. If $|\lambda| < 1$, it is attractive (inwards spiral) and if $|\lambda| > 1$, it is repelling (outwards spiral). #29

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ rotates points by θ around the origin. Depending on θ, it is possible to obtain periodic points of any order. #30

4.3 Iterated Function Systems

Iterated function systems are one of the most common ways to generate fractals. The Cantor Set, Sierpenski Triangle and Gasket are common example of iterated function systems (IFS).

Definition 4.1 ($\mathcal{K}_n$)**.** *Let $\mathcal{K}_n$ be the set of all closed bounded sets in $\mathbb{R}^n$. We say a set $A \subseteq \mathbb{R}^n$ is bounded if there exists an M such that $\forall a \in A, |a| < M$.*

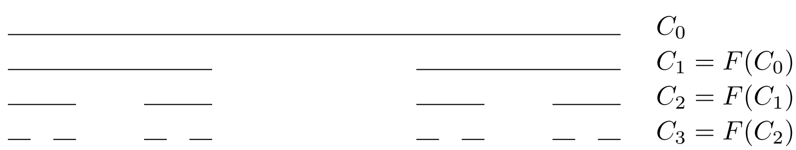
Theorem 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then $f : \mathcal{K}_n \to \mathcal{K}_n$. That is, if A is closed and bounded, so is $f(A) = \{f(a) \mid a \in A\}$.*

Theorem 4.4. *Let $A, B \in \mathcal{K}_n$. Then $A \cup B \in \mathcal{K}_n$.*

Let $C_0 = [0,1] \in \mathcal{K}_1$. Let $f_1(x) = \frac{1}{3}x, f_2(x) = \frac{1}{3}x + \frac{2}{3}$. As both f_1, f_2 are continuous, $f_1(C_0) \cup f_2(C_0) \in \mathcal{K}_1$.

Note that $f_1(C_0) = [0, \frac{1}{3}]$ and $f_2(C_0) = [\frac{2}{3}, 1]$. Let $C_1 = f_1(C_0) \cup f_2(C_0)$.

Now, define $F : \mathcal{K}_1 \to \mathcal{K}_1$ with $F(A) = f_1(A) \cup f_2(A)$. So $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = F(C_0) = F([0,1])$.



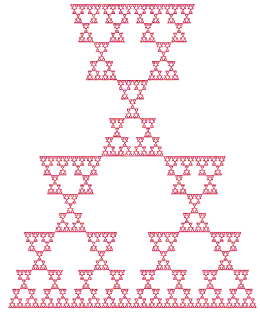
#31

Fact : The Cantor Set is the unique attractive fixed point of F, for $A \in \mathcal{K}_1, A \neq \emptyset$. This is true even if A only contains a single point, and this property holds for all IFS, as we shall prove later.

Now, the concepts of limits and attractiveness need a concept of distance. So we will need to define a distance between sets $A, B \in \mathcal{K}_n$.

Definition 4.2 (Linear contraction)**.** *We say $f(\vec{v}) = A\vec{v} + \vec{b}$ is a linear contraction if there exists $\lambda < 1$ such that $|f(\vec{x}) - f(\vec{y})| < \lambda|\vec{x} - \vec{y}|$ for all $\vec{x}, \vec{y}$.*

Definition 4.3 (Iterated function system)**.** *Let $f_1, ..., f_n$ be linear contractions. Define $F(A) = f_1(A) \cup ... \cup f_n(A)$. We call F an interated function system. F will have a unique attractive fixed point in $\mathcal{K}$, say $A*$. We also call $A*$ an iterated function system.*



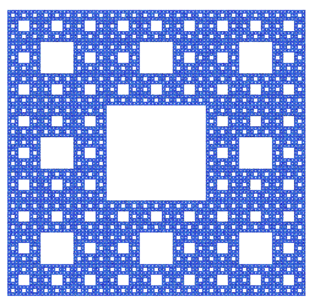
This fractal is composed of three linear contractions (three triangles, one inverted) :

$$f_1(x,y) = (\frac{1}{2}x, \frac{1}{2}y)$$

$$f_2(x,y) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y)$$

$$f_3(x,y) = (\frac{1}{2}x + \frac{1}{4}, 1 - \frac{1}{2}y)$$

#32



This fractal is composed of 8 linear contractions (8 squares) :

$$f_1(x,y) = (\frac{1}{3}x, \frac{1}{3}y) \qquad f_2(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y) \qquad f_3(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$$

$$f_4(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3}) \qquad f_5(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3})$$

$$f_6(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3}) \qquad f_7(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{2}{3}) \qquad f_8(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$$

#33

Definition 4.4 (Distance between closed bounded sets)**.** *Let* $v \in \mathbb{R}^n$*,* $A, B \in \mathcal{K}_n$*. Then :*

$$d(v, B) = \min_{b \in B} |v - b|$$
$$d(A, B) = \max_{a \in A} d(a, B)$$
$$D(A, B) = \max(d(A, B), d(B, A))$$

Notice that the first two distances d are not symmetric while D is. The idea of this definition of distance is that it measures the furthest points from one set to another.

Theorem 4.5 (Properties of D)**.** *Let A, B be non-empty sets.*

1) $D(A,B) = D(B,A)$
2) $D(A,B) \geq 0$
3) $D(A,B) = 0$ iff $A = B$
4) $D(A,B) \leq D(A,C) + D(C,B)$

Proof. (4) (Triangle Inequality) : Notice that $d(a,B) \leq |a-b|\ \forall\ b \in B$.

$$\begin{aligned} d(a,B) &\leq |a-c| + |c-b| \quad \forall\ b \in B, c \in C \\ &\leq |a-c| + d(c,B) \quad \forall\ c \in C \\ &\leq d(a,C) + d(C,B) \\ &\leq d(A,C) + d(C,B) \\ &\leq D(A,C) + D(C,B) \end{aligned}$$

So $d(a,B) \leq D(A,C) + D(C,B)$, from which we get $d(A,B) \leq D(A,C) + D(C < B)$. Similarly, $d(B,A) \leq D(A,C) + D(C,B)$.

$\therefore D(A,B) \leq D(A,C) + D(C,B)$. ☐

Definition 4.5 (Limits of sequence of sets)**.** *We say $\lim_{n\to\infty} A_n = A*$ if $\lim_{n\to\infty} D(A_n, A*) = 0$.*

Theorem 4.6 (Convergence of IFS)**.** *For an IFS, there exists an unique attractive fixed point $A* \in \mathcal{K}_n$ such that :*

1) $F(A*) = A*$
2) $\lim_{n\to\infty} F^{[n]}(A) = A*$ for all $A \in \mathcal{K}_n$

This proof is fairly long, so we present the outline of the proof as follows :

1) Define a Cauchy Sequence
2) Show that $\forall\ A$, $\{F^{[n]}(A)\}$ is a Cauchy Sequence
3) Define a complete metric space and claim $\mathcal{K}_n$ with D is a complete metric space.
4) We will observe that $F(A*) = A*$ and that $\lim F^{[n]}(A) = A*$ exists.
5) Show that limits are unique - i.e., $\lim F^{[n]}(A) = \lim F^{[n]}(B)$

Definition 4.6 (Cauchy sequence)**.** *We say a sequence A_n is a Cauchy sequence if $\forall\ \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n, m \geq N$, then $D(A_n, A_m) < \epsilon$.*

Let $\{\frac{1}{n}\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ with distance $D(x,y) = |x-y|$. We claim that this is a Cauchy sequence. #34

Simply take $N = \lceil \frac{1}{\epsilon} \rceil$ and see if $n, m \geq N$, then $|\frac{1}{n} - \frac{1}{m}| < \epsilon$.

Let $\{C_n\}$ be the sequence of intervals going to the Cantor Set. We had

$$D(C_n, C_m) = \frac{1}{2 \cdot 3^{\min(n,m)+1}}$$

For any $\epsilon > 0$, there will exist an N such that $\frac{1}{2 \cdot 3^{N+1}} < \epsilon$. So for $n, m > N$, we will have #35

$$D(C_n, C_m) = \frac{1}{2 \cdot 3^{\min(n,m)+1}} \leq \frac{1}{2 \cdot 3^{N+1}} < \epsilon$$

This proves the result.

Lemma 4.7. *Let f be a linear contraction such that $|f(\vec{x}) - f(\vec{y})| < \lambda|\vec{x} - \vec{y}|$. Then for all $A, B \in \mathcal{K}_n$, $D(f(A), f(B)) < \lambda D(A, B)$.*

Proof.

$$d(f(a), f(B)) = \min_{b \in B} |f(a) - f(b)| < \lambda \min_{b \in B} |a - b| < \lambda d(a, B)$$
$$d(f(A), f(B)) = \max_{a \in A} d(f(a), f(B)) < \lambda \max_{a \in A} d(a, B) < \lambda d(A, B)$$

Similarly, $d(f(B), f(A)) < \lambda d(B, A)$.

$\therefore D(f(A), f(B)) < \lambda D(A, B)$ by combining the two.

□

Lemma 4.8.

$$D(A_1 \cup A_2, B_1 \cup B_2) \leq \max(D(A_1, B_1), D(A_2, B_2))$$

Intuitively, this is true because the distance D is defined as the maximum distance between points in two sets. Therefore the distance between two large sets ought to be smaller than the distance between two small sets.

Proof. Note $D(A_1 \cup A_2, B_1 \cup B_2) = \max(d(A_1 \cup A_2, B_1 \cup B_2), d(B_1 \cup B_2, A_1 \cup A_2))$. Consider

$$\begin{aligned} d(A_1 \cup A_2, B_1 \cup B_2) &= \max_{a \in A_1 \cup A_2} d(a, B_1 \cup B_2) \\ &= \max(\max_{a \in A_1} d(a, B_1 \cup B_2), \max_{a \in A_2} d(a, B_1 \cup B_2)) \\ &\leq \underbrace{\max(\max_{a \in A_1} d(a, B_1), \max_{a\ in A_2} d(a, B_2))}_{\text{this works because d is defined as a minimum}} \\ &= \max(d(A_1, B_1), d(A_2, B_2)) \end{aligned}$$

Similarly, $d(B_1 \cup B_2, A_1 \cup A_2) \leq \max(d(B_1, A_1), d(B_2, A_2))$.

Combining these gives $D(A_1 \cup A_2, B_1 \cup B_2) \leq \max(D(A_1, B_1), D(A_2, B_2))$.

□

Corollary 4.1. *Let $F : \mathcal{K}_n \to \mathcal{K}_n$ with $F(A) = f_1(A) \cup f_2(A) \cup ... \cup f_k(A)$ with the property that $|f_i(x) - f_i(y)| < \lambda|x - y|$ for all x, y and fixed $\lambda < 1$.*

Then $D(F(A), F(B)) < \lambda D(A, B)$. Further, $D(F^{[l]}(A), F^{[l]}(B)) < \lambda^l D(A, B)$.

Proof.

$$\begin{aligned}
D(F(A), F(B)) &= D(f_1(A) \cup f_2(A) \cup ... \cup f_k(A), f_1(B) \cup f_2(B) \cup ... \cup f_k(B)) \\
&\leq \max(D(f_1(A), f_1(B)), \max(D(f_2(A), f_2(B)), ..., \max(D(f_1(A), f_k(B))) \\
&\leq \max(\lambda D(A, B), ..., \lambda D(A, B)) \\
&= \lambda D(A, B)
\end{aligned}$$

□

Theorem 4.9. *Let $\left\{F^{[n]}(A)\right\}_{n=0}^{\infty}$ be a sequence in $\mathcal{K}_n$. Then this is a Cauchy sequence.*

Proof. Let $\lambda < 1$ be the contraction upper bound as defined earlier. Let $k = D(F(A), A)$. Then

$$D(F(A), F^{[2]}(A)) \leq \lambda k$$
$$D(F^{[m]}(A), F^{[m+1]}(A)) \leq \lambda^m k$$

Assume w.l.o.g. $m \leq n$. Then we apply the triangle inequality multiple times :

$$\begin{aligned}
D(F^{[m]}(A), F^{[n]}(A)) &\leq D(F^{[m]}(A), F^{[m+1]}(A)) \\
&\quad + D(F^{[m+1]}(A), F^{[m+2]}(A)) \\
&\quad + ... \\
&\quad + D(F^{[n-1]}(A), F^{[n]}(A)) \\
&\leq \lambda^m k + \lambda^{m+1} k + ... + \lambda^{n-1} k \\
&= \lambda^m (1 + \lambda + \lambda^2 + ... + \lambda^{n-m-1}) k \\
&\leq \lambda^m (1 + \lambda + \lambda^2 + ...) k \\
&= \lambda^m k \left(\frac{1}{1-\lambda}\right)
\end{aligned}$$

As $\lambda < 1$, we can pick N such that $\lambda^N k \left(\frac{1}{1-\lambda}\right) < \epsilon$. Hence, for all $n, m \geq N$, we have $D(F^{[n]}(A), F^{[m]}(A)) \leq \lambda^N k \left(\frac{1}{1-\lambda}\right) < \epsilon$.

This shows that $\left\{F^{[n]}(A)\right\}_{n=0}^{\infty}$ is a Cauchy sequence.

□

Definition 4.7 (Complete metric space)**.** *We say a metric space is complete if every Cauchy sequence converges.*

In other words, if $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence, then $\lim_{n\to\infty} A_n$ exists and is in the space.

$\mathbb{R}$ is a complete metric space, $\{0, 1\}$ and $\mathbb{Q}$ are not complete.	#36

As this is more complicated and beyond the scope of this course, we shall state as a fact that $\mathcal{K}_n$ is also a complete metric space.

Corollary 4.2. $\lim\limits_{n\to\infty} F^{[n]}(A)$ *exists and is in* $\mathcal{K}_n$*.*

Corollary 4.3. *Let* $A* = \lim\limits_{n\to\infty} F^{[n]}(A)$*. Then* $F(A*) = A*$*.*

Proof. The proof is the same as Theorem 1.1.

□

Corollary 4.4. *Let* $A* = \lim\limits_{n\to\infty} F^{[n]}(A)$ *and* $B* = \lim\limits_{n\to\infty} F^{[n]}(B)$*. Then* $A* = B*$*.*

Proof. $D(A*, B*) = D(F(A*), F(B*)) < \lambda D(A*, B*)$. This has only one solution, $D(A*, B*) = 0 \implies A* = B*$.

□

4.4 Drawing IFS

We know that $F^{[n]}(A) \to A*$, so we can start with a nice value for A, say a square or a circle, and iterate $F^{[n]}(A)$ for large n in order to get the picture of the fractal we want.

However, this has a problem. If $F(A) = f_1(A) \cup f_2(A) \cup ... \cup f_k(A)$ is made up of k maps, then $F^{[n]}(A)$ is made up of k^n maps. This would give an exponential runtime for the drawing algorithm.

Easier monte carlo method : Take a point, preferably but not neccessarily a fixed point of f_i, call this $\vec{v_1}$. Pick a f_i at random from $\{f_1, f_2, ..., f_k\}$. Compute and draw $\vec{v_2} = f_i(\vec{v_1})$ and repeat.

See `http://en.wikipedia.org/wiki/Chaos_game` for Sierpenski triangle animation.	#37

Sometimes we do not want to choose all maps with the same probability. If the maps have different contractions, we will want the maps that contract more to have a lower probability, to keep the same density.

While it would be possible to formally determine the optimal ratios, in practice it is sufficient to take a guess and adjust as needed.

4.5 Open set condition

Definition 4.8 (Open set condition). *We say $F : \mathcal{K}_n \to \mathcal{K}_n$ satisfies the open set condition if there exists an open set J such that :*
1) $J \supseteq f_1(J) \cup f_2(J) \cup ... \cup f_k(J)$
2) $f_i(J) \cap f_j(J) = \emptyset$ for $i \neq j$

In other words, an iterated function satisfies the open set condition if its components at every iteration do not overlap.

Theorem 4.10. *Let F satisify the open set condition, $F(A) = f_1(A) \cup f_2(A) \cup ... \cup f_k(A)$. Further, let μ_i, λ_i satisfy $\mu_i|x - y| \leq |f_i(x) - f_i(y)| \leq \lambda_i|x - y|$.*

Let d_1, d_2 satisfy $\mu_1^{d_1} + \mu_2^{d_2} + ... + \mu_k^{d_k} = 1$ and $\lambda_1^{d_1} + \lambda_2^{d_2} + ... + \lambda_k^{d_k} = 1$

Then $d_1 \leq$ dimension of the fractal $\leq d_2$.

5 Complex Fractals

As we begin to study iterated functions over the complex plane, it is useful to recall the Fundamental Theorem of Algebra.

Theorem 5.1 (Fundamental Theorem of Algebra)**.** *A polynomial $p(z) \in \mathbb{C}[x]$ of degree n can be written as $p(z) = a(z - \alpha_1)(z - \alpha_2)...(z - \alpha_n)$, α_i not necessarily distinct. In other words, p always has n roots.*

5.1 Newton Iterates

Recall that a differentiable function f with root at p, we define the Newton Iterate as $g(z) = z - \frac{f(z)}{f'(z)}$.

If p is a root of $f(z)$, then it is an attractive fixed point of $g(z)$. This works for complex functions.

Definition 5.1 (Basin of Attraction)**.** *Let p be a root of $f(z)$, hence an attractive fixed point of $g(z)$. We define the Basin of Attraction of p as $\{z \mid g^{[n]}(z) \to p \text{ as } n \to \infty\}$.*

Let $f(z) = z^2 - 1$, which has roots at $z = \pm 1$. Here $g(z) = z - \frac{f(z)}{f'(z)} = z - \frac{z^2-1}{2z} = z - \frac{z}{2} + \frac{1}{2z} = \frac{1}{2}(z + \frac{1}{z})$.

Which z are in the basin of attraction of 1?

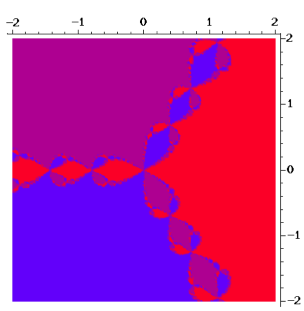
We see that for $x \in \mathbb{R}, x > 0$ that $g^{[n]}(x) > 1$ (the function has a local minimum at $x = 1$).

For larger number, take for example $z_1 = 10000(1 + i)$. Then $z_2 \approx 5000(1 + i)$. In general, for large numbers, $g(z) \approx \frac{1}{2}z$. #38

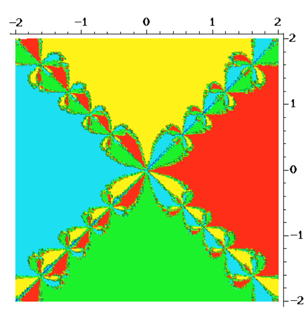
What if $|z| = 1$, $z \neq \pm 1$ or $\pm i$? Then $g(z) = \frac{1}{2}(z + \frac{1}{z}) = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}\Re(2z) = \Re(z)$.

So if $\Re(z) > 0$, then $g^{[n]}(z) \to 1$ as we know what happens to iterates in $\mathbb{R}$.

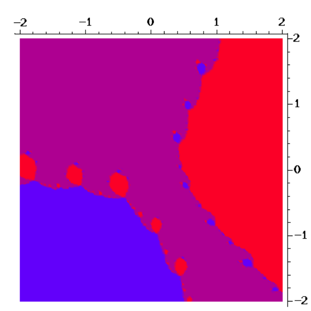
Below are some more examples of basins of attraction (each color is a different basin).



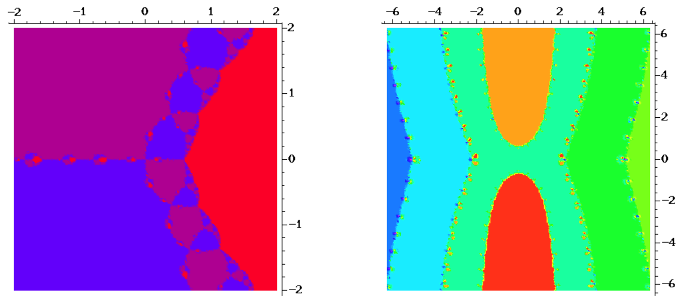
$z^3 - 1$



$z^4 - 1$



$z^3 - zi - 1$



$z^3 - z - 1$ $\sin(z)(z^2+1)$

Figure 1: Various basins of attractions

5.2 Julia Sets

Consider $g_c(z) = z^2 + c$ as a map from $\mathbb{C} \to \mathbb{C}$ and $c \in \mathbb{C}$. As before, we wish to examine iterates of this function, finding fixed, periodic points and when they are attractive or repelling.

Iterates of $g_c(0)$ for various c.

$$\{{g_0}^{[n]}(0)\} = \{0, 0, 0, ...\}$$

$$\{{g_1}^{[n]}(0)\} = \{0, 1, 2, 5, 26, ...\}$$

$$\{{g_{-1}}^{[n]}(0)\} = \{0, -1, 0, -1, ...\}$$ #39

$$\{{g_i}^{[n]}(0)\} = \{0, i, -1+i, -i, -1+i, -i, ...\}$$

$$\{{g_{2i}}^{[n]}(0)\} = \{0, 2i, -4+2i, 12-14i, 52-334i, ...\}$$

Consider $g_i(z) = z^2 + i$. Notice $g_i(z) = z \implies z^2 + i - z = 0$ which has roots by the fundamental theorem of algebra. These are $z = \frac{1 \pm \sqrt{1-4i}}{2} \approx 1.3 - 0.6i, -0.3 + 0.6i$. Notice that since $g'(z) = 2z$, both of these roots are repelling. #40

In the previous example, we saw also that $\{-i, -1+i\}$ was a two-cycle.

Find all periodic points of $g_0(z) = z^2 + 0$ and classify.

The fixed points of g_0 are 0, which is attractive, and 1, which is repelling. #41

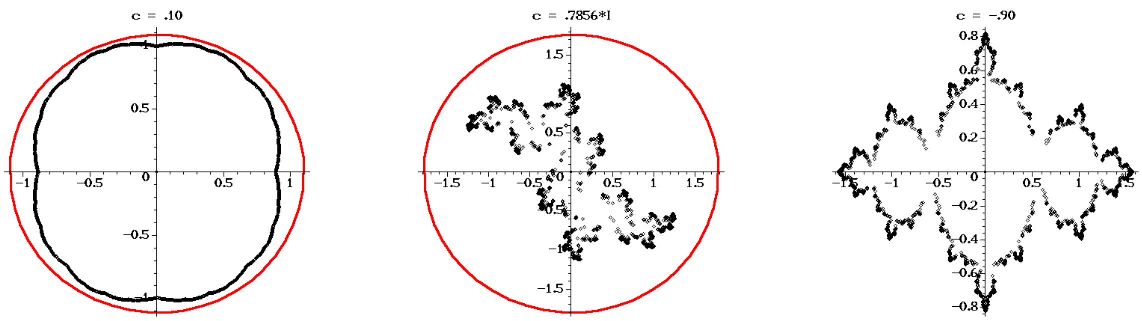
In general, $g^{[n]}(z) = z^{2^n} = z \implies z^{2^n - 1} = 1$ gives roots of the form $z = e^{\frac{2\pi i \cdot j}{2^n - 1}}, j = 0, 1, ..., 2^n - 1$.

These form a circle of radius 1.

Definition 5.2 (Julia set)**.** *The Julia set of $g_c(z) = z^2 + c$ denoted J_c is the smallest closed set containing all the repelling periodic points of $g_c(z)$.*

$J_0 = \{z \mid |z| = 1\}$ as seen earlier. #42

Below are some examples of Julia sets.



$c = 0.1$ $c = 0.7856i$ $c = -0.90$

Figure 2: Julia sets for various c.

Theorem 5.2. *If $|z| > |c| + 1$, then $z \notin J_c$.*

Proof. Let $z_0 = z, z_1 = g_c(z), ..., z_n = {g_c}^{[n]}(z_0)$.

$$\begin{aligned} |z_1| = |g_c(z_0)| = |z_0^2 + c| &= |z_0||z_0 + \frac{c}{z_0}| \\ &> |z_0|(|z_0| - |\frac{c}{z_0}|) \\ &> |z_0|(|z_0| - \frac{|c|}{|c|+1}) \\ &> |z_0|(|c| + 1 - |c|) = |z_0| \end{aligned}$$

Repeating this gives $|c| + 1 < |z_0| < |z_1| < |z_2| < ....$

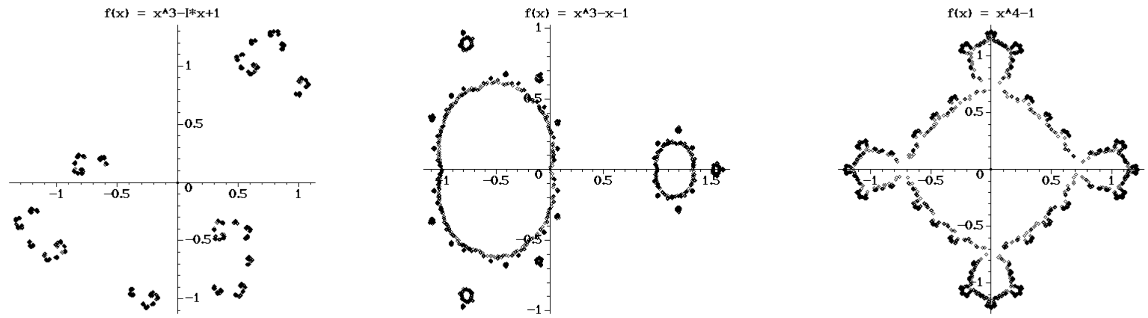
Hence z_0 is not a periodic point, therefore $z \notin J_c$.

Note that it is possible to use this proof to show that $\lim_{n\to\infty} |z_n| = \infty$.

□

Definition 5.3. *Let $p(z) \in \mathbb{C}[z]$ be any polynomial. We define $J_{p(z)}$ as the Julia set associated with $p(z)$ as the smallest closed set containing all repelling periodic points of $p(z)$.*

More examples below.



$p(z) = x^3 - ix + 1$ $\qquad$ $p(z) = x^3 - x - 1$ $\qquad$ $p(z) = x^4 - 1$

Figure 3: Julia sets for various $p(z)$

As a fact, just like the regular Julia set, $J_{p(z)}$ is bounded.

Other facts about the Julia set :

1. Julia sets have an uncountable number of points.
2. The set is closed and bounded.
3. The set contains no isolated points. That is, for all $z \in J_c$, there exists a sequence $z_n \in J_c, z_n \neq z$ such that $\lim z_n = z$.
4. $f(J_{f(z)}) = f^{-1}(J_{f(z)}) = J_{f(z)}$.
5. If $z \in J$, then J is the smallest closed set containing $\displaystyle\bigcup_{k=0}^{\infty} f^{-k}(z)$.
6. The function $f(z)$ is transitive on J.
7. The Julia set is the boundary of the basin of attraction of each attractive fixed/periodic point.
8. Similarly, the Julia set is the boundary of the basin of attraction of ∞.
9. $J_{f(z)} = J_{f^{[n]}(z)}$.

The property that J_c is the boundary of the basin of attraction of infinity gives us a method to draw J_c.

Method 1 : Take a point in some range $x \in [a, b], y \in [c, d], z = x + yi$. Iterate the point. If it is eventually bigger than $|c| + 1$, it is outside J_c. If after some number of iterations, it has still remained bounded, then it is probably inside J_c.

By controlling the number of iterations, we can control the accuracy of our method. Furthermore, if we draw the inside and outside with different colors, we will get the boundary.

Method 2 : Using the fact that J_c is the smallest closed set containing $\bigcup_{k=0}^{\infty} f^{-k}(z)$ gives us a second method. Start with a point on J and notice that $f^{-1}(z)$ has multiple solutions. Pick one at random and plot it (or all solutions found) and repeat on a random point.

Theorem 5.3. *If p is an attractive fixed or periodic point of any polynomial $f(z)$, then at least one of the critical points of $f(z)$ lies in the basin of attraction of p.*

Corollary 5.1. *The function $g_c(z) = z^2 + c$ has at most one attractive fixed point or periodic point in which 0 is in the basin of attraction for this attractive fixed/periodic point.*

Theorem 5.4. *If ${g_c}^{[n]}(0) \to p$ where p is an attractive fixed point, the J_c is a simple closed curve - i.e., it does not ever come in contact with or intersect itself.*
If ${g_c}^{[n]}(0) \to p$ where p is an attractive periodic point, the J_c is a closed curve, but not simple.
If ${g_c}^{[n]}(0) \to \infty$, then J_c is totally disconnected. This is sometimes called Fatou dust.

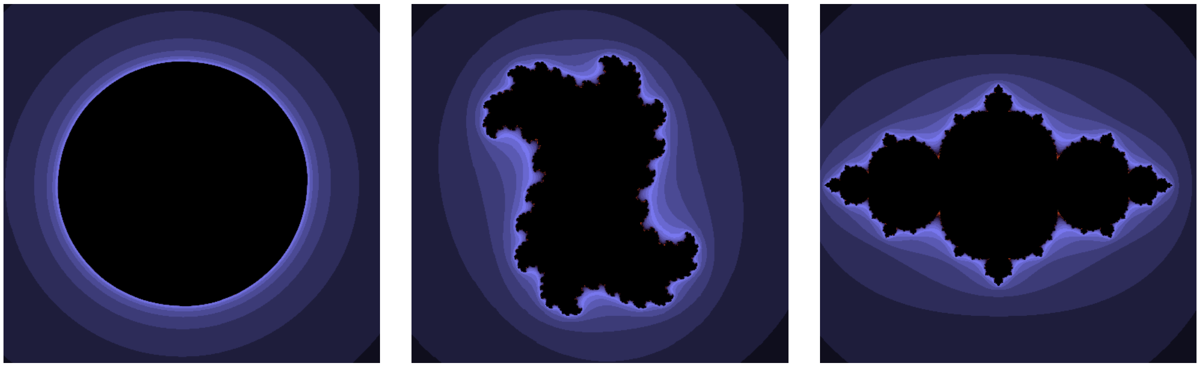


Figure 4: Julia sets associated with an attractive fixed point

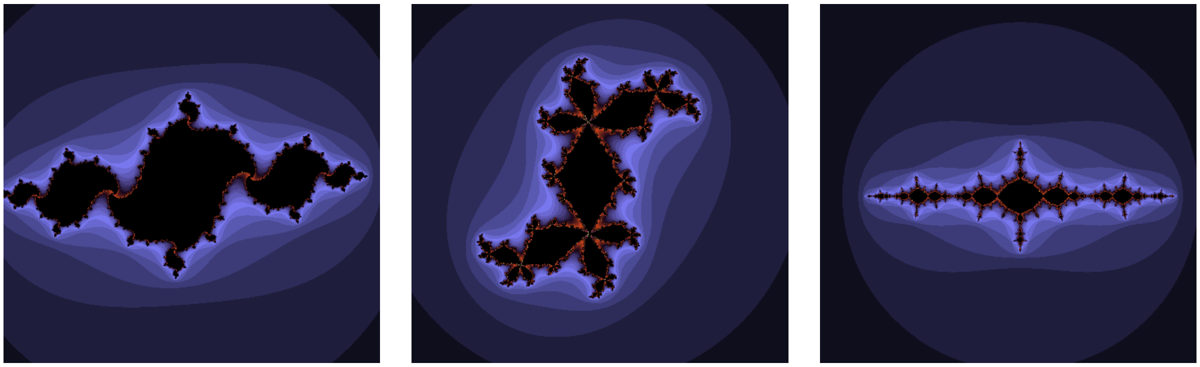


Figure 5: Julia sets associated with an attractive periodic point

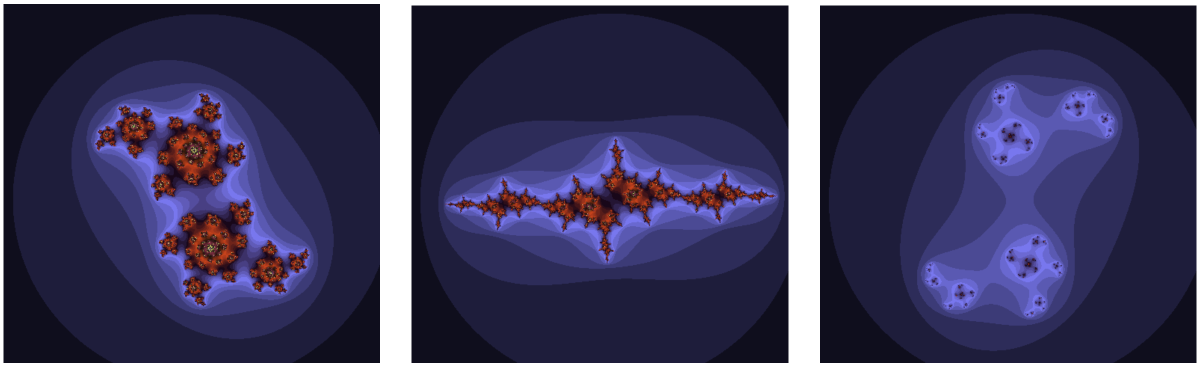


Figure 6: Julia sets associated with infinity.

Theorem 5.5. *Suppose* $|c| > \frac{1}{4}(5+\sqrt{6}) \approx 2.474$*, then* J_c *is totally disconnected and for large* c*,* $\dim_c(J_c) \approx 2\frac{\log 2}{\log|c|}$

Proof. We will show the stronger result that if $|c| > 2$, then J_c is totally disconnected. It suffices to show if $|c| > 2$, then ${g_c}^{[n]} \to \infty$.

Note that $g_c(0) = c, {g_c}^{[2]} = c^2+c, ..., {g_c}^{[n]}(0) = {g_c}^{[n-1]}(0)^2+c$. We will claim by induction that $|{g_c}^{[n]}(0)| \geq |c|$ for $n \geq 1$.

This is clearly true for $n = 1$. For the inductive case,

$$\begin{aligned}
|{g_c}^{[n]}(0)| &= |{g_c}^{[n-1]}(0)^2 + c| \\
&\geq |{g_c}^{[n-1]}(0)|^2 - |c| && \text{by triangle inequality} \\
&\geq |c||{g_c}^{[n-1]}(0)| - |{g_c}^{[n-1]}(0)| && \text{by inductive assumption} \\
&= (|c|-1)|{g_c}^{[n-1]}(0)| \\
&\geq (|c|-1)^2|{g_c}^{[n-2]}(0)| && \text{using the same sequence of steps} \\
&\geq ... \\
&\geq (|c|-1)^{n-1}|g(0)| \\
&= (|c|-1)^{n-1}|c|
\end{aligned}$$

As $|c| > 2$, we have $|c| - 1 > 1$, hence $|{g_c}^{[n]}(0)| \geq |c|$ and $|{g_c}^{[n]}(0)| \to \infty$ as $n \to \infty$.

Now we can prove that $\dim(J_c) \approx 2\frac{\log 2}{\log|c|}$. To do so, we will 1) write J_c as an IFS, 2) show that this satifies the open set condition, 3) determine the contraction ratio for maps - i.e., $\mu|x-y| \leq |f(x) - f(y) \leq \lambda|x-y|$ and 4) use the dimension to formula seen earlier.

First we show that J_c is an IFS. Note $J_c = g_c(J_c) = g_c^{-1}(J_c) \implies g_c^{-1}(z)$ has two values, $f_1(z) = \sqrt{z-c}$ and $f_2(z) = -\sqrt{z-c}$.

So $J_c = g_c^{-1}(J_c) = f_1(J_c) \cup f_2(J_c)$. We will show later that these are contractions. For now, we will show this satisfies the OSC.

i.e., We will need to find an open set C such that $C \supseteq f_1(C) \cup f_2(C)$ and $f_1(C) \cap f_2(C) = \emptyset$.

Let $C = \{z \mid |z| < |c|\}$.

As $|z| < |c|$, we have $|\sqrt{-2c}| < \sqrt{|c^2|} = |c|$.

So we have $f_1(C) \cup f_2(C) \subseteq C$, $f_1(C) \cap f_2 C = \emptyset$ so this satifies the OSC.

Here, C shows it satifies the OSC, but we will need a stronger result for later. Note that the point of maximal distance in $f_1(C)$ is $\approx \sqrt{-2c}$.

So define $V = \{z \mid |z| < \sqrt{2|c|}\}$. This again satifies the OSC.

Our goal now is to show f_1 and f_2 look like linear contractions for large c. Consider two points $z, w \in V$ close to each other. Then

$$\frac{f_1(z) - f_1(w)}{z - w} \approx f'(z) = \frac{1}{2\sqrt{z-c}}$$

This implies that $|f_1(z) - f_1(w)| \approx |f_1'(z)||z - w|$.

$$\begin{aligned}
|f'(z)| &= |\frac{1}{2\sqrt{z-c}}| \\
&= \frac{1}{2\sqrt{|z-c|}} \\
&= \frac{1}{2\sqrt{|c-z|}} \\
&\leq \frac{1}{2\sqrt{|c| - |z|}} \\
&\leq \frac{1}{2\sqrt{|c| - \sqrt{2|c|}}} && \text{since} |z| < \sqrt{2|c|}
\end{aligned}$$

We will want $|f_1'(z)| < 1$ for this to be a contraction, which will happen if

$$\begin{aligned}
&\frac{1}{2\sqrt{|c| - \sqrt{2|c|}}} < 1 \\
&\implies 2\sqrt{|c| - \sqrt{2|c|}} > 1 \\
&\implies 4(|c| - \sqrt{2|c|}) > 1 \\
&\implies 4|c| - 1 > 4\sqrt{2|c|} \\
&\implies 16|c|^2 - 8|c| + 1 > 32|c| \\
&\implies 16|c|^2 - 40|c| + 1 > 0 \\
&\implies |c| > \frac{5 + 2\sqrt{6}}{4} \approx 2.475
\end{aligned}$$

So if $|c| > \frac{5+2\sqrt{6}}{4}$, then $|f_1'(z)| < 1$ and we have a linear contraction. We can similarly show that $|f_1'(z) > \frac{1}{2\sqrt{|c| + \sqrt{2|c|}}}$. Hence, for sufficiently large $|c|$, we have $|f'(z)| \approx \frac{1}{2\sqrt{|c|}}$. Therefore, we will have that $|f_1(z) - f_1(w)| \approx \frac{|z-w|}{|2\sqrt{|c|}}$. Similarly for f_2.

Further, we know $J_c = f_1(J_c) \cup f_2(J_c)$ satifies the open set condition. Hence the dimension must satisfy:

$$\begin{aligned}
&\left(\frac{1}{2\sqrt{|c|}}\right)^d + \left(\frac{1}{2\sqrt{|c|}}\right)^d = 1 \\
&\implies d \log \frac{1}{2\sqrt{|c|}} = \log \frac{1}{2} \\
&\implies d \log 2\sqrt{|c|} = \log 2 \\
&\implies d = \frac{\log 2}{\log 2\sqrt{|c|}} = \frac{\log 2}{\log 2 + \frac{1}{2}\log|c|} \approx \frac{2\log 2}{\log|c|} && \text{with large } |c|
\end{aligned}$$

This is the desired result. □

5.3 Mandelbrot Set

Recall that :

1. If ${g_c}^{[n]} \to p$ where p is an attractive fixed point, the J_c is a simple closed curve.
2. If ${g_c}^{[n]} \to p$ where p is an attractive periodic point, the J_c is a closed curve, but not simple.
3. If ${g_c}^{[n]} \to \infty$, then J_c is totally disconnected.

We will use this to define M, the Mandelbrot set, as $M = \{c \mid {g_c}^{[n]}(0) \not\to \infty\}$.

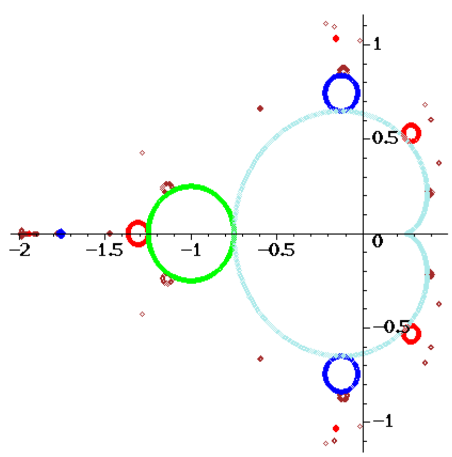


Figure 7: Regions of the Mandelbrot set, as they relate to Julia sets.

A good website with a "guide" to the patterns of the Mandelbrot and their associated Julia Set can be found here `http://www.miqel.com/fractals_math_patterns/visual_math_varieties.html`

Find all c such that ${g_c}^{[n]} \to p$ where p is an attractive periodic point of period 2. It suffices to show for which c does $g_c(0)$ have an attractive periodic point of period 2?

This is true because we know 0 is the basin of attraction.

For fixed points, $g_c(z) = z \implies z^2 + c = z \implies z^2 - z + c = 0$. So if $z^2 - z + c = 0$, then z is a fixed point. We don't want these. For points of periods 2, we need

$$\begin{aligned}
&g_c(g_c(z)) = z \\
&\implies (g_c(z))^2 + c = z \\
&\implies (z^2 + c)^2 + c = z \\
&\implies z^4 + 2cz^2 - z + c + c^2 = 0 \\
&\implies \underbrace{(z^2 - z + c)}_{\text{fixed points, not wanted}} \underbrace{(z^2 + z + c + 1)}_{\text{period 2 points, wanted}} = 0
\end{aligned}$$

#43

So if $z^2 + z + c + 1 = 0$, then we have a periodic point of period 2. Equivalently,

$$z = \frac{-1 \pm \sqrt{1 - 4c - 4}}{2} \implies z = \frac{-1 \pm \sqrt{-4c - 3}}{2}$$

We want $|({g_c}^{[2]})'(z)| < 1$. We can expand this expression:

$$\begin{aligned}
({g_c}^{[2]})'(z) &= (z^4 + 2cz^2 + c + c^2)' \\
&= 4z^3 + 4cz \\
&= (4z - 4)\underbrace{(z^2 + z + c + 1)}_{\text{0 for periodic points}} + 4 + 4c \\
&= 4 + 4c
\end{aligned}$$

This gives that $g_c(z)$ has an attractive fixed point of period 2 if $|4c + 4| < 1$ or, equivalently, $|c + 1| < \frac{1}{4}$. This is a circle of radius $\frac{1}{4}$ centered at -1.

If there is a birfurcation point between two parts of the Mandelbrot Set (i.e., that associated with attractive periodic points of period n, and those with period m), then $n|m$ or $m|n$. This corresponds to where circles touch in the previous image.

Theorem 5.6 (Escape Radius). *If $|c| > 2$, the escape radius, then $c \notin M$. Furthermore, if there exists a k such that $|z_k| = |{g_c}^{[k]}(0)| > 2$, then c is not in the Mandelbrot Set.*

Proof. The fact that $|c| > 2 \implies c \notin M$ follows from what we have already proved about the Julia Set. If $|c| > 2$, then $|{g_c}^{[n]}(0)| \to \infty$ and J_c is dust-like, which shows $c \notin M$.

For the proof of the second statement, assume $|z_k| = |{g_c}^{[k]}(0)| > 2$. We can assume w.l.o.g that $|c| \leq 2$, otherwise we'd be done already. Hence, $|z_k| > |c|$. Pick $n \geq k$. We will claim that $|z_n| > |z_k|$ by induction.

$$
\begin{aligned}
|z_n| &= |g_c(z_{n-1})| \\
&= |z_{n-1}^2 + c| \\
&\geq |z_{n-1}^2| - |c| \\
&\geq |z_k||z_{n-1}| - |z_{n-1}| \\
&= (|z_k| - 1)|z_{n-1}| \\
&> |z_{n-1}| && \text{since } |z_k| > 2
\end{aligned}
$$

Further, $|z_n| \geq (|z_k| - 1)|z_{n-1}| \geq (|z_k| - 1)^2|z_{n-2}| \geq ... \geq (|z_k| - 1)^{n-k}|z_k|$, so $|z_n| \to \infty$ as required.

$\square$

Having an escape radius allows for an algorithm to draw M. For each c in some region of $\mathbb{C}$, iterate ${g_c}^{[n]}(0)$ for some large n. If at any point, $|{g_c}^{[k]}(0)| > 2$, then we know $c \notin M$. We usually color this point based on k, the first time $|{g_c}^{[k]}(0)| > 2$. This is often referred to as escape-time coloring.

If after n iterates, we still have $|{g_c}^{[n]}(0)| > 2$, we are probably in M and we color the point black.

Recall : For large c, we know that $\dim(J_c) \approx \frac{2 \log 2}{\log |c|}$.

Fact : For small c, we have $\dim(J_c) = 1 + \frac{|c|^2}{4 \log 2} + O(|c|^3)$. Let ∂M be the boundary of M (i.e., if $c \in \partial M$, there are points arbitrarily close to c in M and points arbitrarily close to c not in M. Then it has been proven that $\dim(\partial M) = 2$. That is, the boundary of the Mandelbrot set is space-filling. Further, if $c \in \partial M$, then $\dim(J_c) = 2$.

6 Infinite Binary Trees

6.1 Binary trees and self-contacting points

We define $T(r_1, r_2, \theta_1, \theta_2)$ as the infinite binary fractal tree. It consists of a line from $(0, 0)$ to $(0, 1)$ and a scaled copy of T (by a factor r_1 at angle θ_1) attached to $(0, 1)$ and a second scaled copy (by r_2 at angle θ_2) attached in the same place.

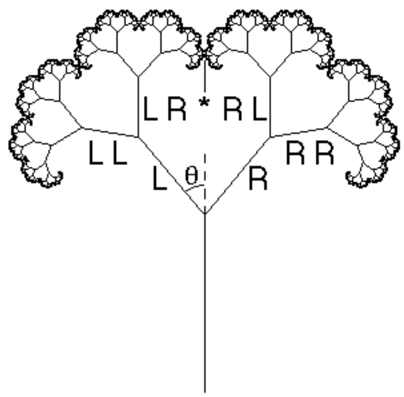


Figure 8: Reference diagram, from http://www.math.union.edu/research/fractaltrees/FractalTreesDefs.html

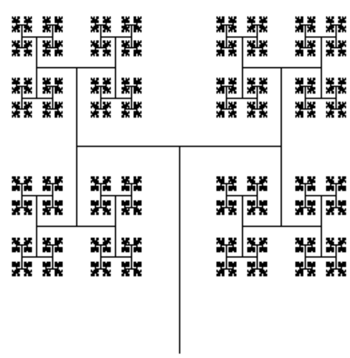


Figure 9: $T(\frac{1}{2}, \frac{1}{2}, 90°, 90°)$

#44

$T(1, \frac{1}{2}, 120°, -120°)$
This one actually becomes a sierpenski triangle, but is an extremely odd binary tree.

#45

There are three common types of trees.

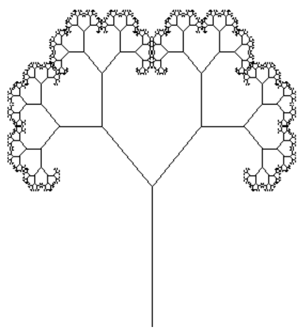
Definition 6.1 (Overlapping). *We say a tree is overlapping if two of the branches cross.*

Definition 6.2 (Self-avoiding). *We say a tree is self-avoiding if there is a unique path from every point in T to any other point.*

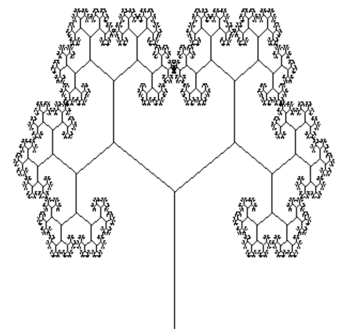
Self-avoiding also means that the tree has no loop.

Definition 6.3 (Self-contacting). *A tree is self-contacting if it is not overlapping and not self-avoiding.*

Our main question is, when do each of these three cases occur?



$\theta = 45°$ Self-contacting at multiple points.



$\theta = 60°$ Self-contacting at a single point.

Definition 6.4 (Symmetric). *A symmetric infinite binary fractal tree has $T(r, \theta) = T(r, r, \theta, \theta)$.*

In this section, we will focus our attention on symmetric infinite binary fractal trees. We would like to find out what values r give self-contacting for a given θ, when trees can be space-filling and what can be said about the dimension of $T(r, \theta)$.

To do so, we need to introduce some notation. Denote by R the end of the rightmost branch from $(0, 1)$ and L, the left (see reference diagram).

We will define by $w_1 w_2 ... w_n$, $w_i \in \{R, L\}$, the $w_2 w_3 ... w_n$ branch of the w_1 branch. By an infinite word $w_1 w_2 w_3 ...$, we mean the "leaf" associated with $\lim_{n\to\infty} w_1 w_2 ... w_n$. Also, $(RL)^\infty = RLRLRL...$

Theorem 6.1. *A self-avoiding tree has an uncountable number of leaves of infinite words.*

Given some infinite word, we can compute its precise x and y values.

For $T(\frac{1}{2}, 90°)$, compute the coordinates of RL and R^∞.

Since RL is a finite word, it can easily be computed as $(0+r\sin(90°)+r^2\sin(0°), 1+r\cos(90°)+r^2\cos(0°)) = (\frac{1}{2}, \frac{5}{4})$.

Consider $RRR... = R^\infty$. Then the x-coordinate is

$$
\begin{aligned}
&0 + r\sin(90°) + r^2\sin(180°) + r^3\sin(270°) + r^4\sin(270°) + r^4\sin(0°) \\
&= r - r^3 + r^5 - r^7 + r^9 - ... \\
&= (1 - r^3)(1 + r^4 + r^8 + r^12 + ...) \\
&= \frac{r - r^3}{1 - r^4} = \frac{\frac{1}{2} - \frac{1}{8}}{1 - \frac{1}{16}} = \frac{3/8}{15/16} = \frac{2}{5}
\end{aligned}
$$

We can similarly find the y-coordinate is $\frac{4}{5}$.

#46

Theorem 6.2. *If a tree is self-contacting, then there exists a word (finite or infinite) such that it has x-coordinate 0.*

Proof. For a tree to be self-contacting, the tip of some branch (possibly infinite) will touch the tip or the interior of some other branch.

We can assume w.l.o.g. that one of these starts with L and the other with R. If both started with L, then they would both be in the left copy of the original tree which is identical to the original tree.

We now take these two branches, and replace every L with R, and every R with L. This gives a second self-contacting point. This new point has coordinate $(-x, y)$ if the first has coordinate (x, y). If $x \neq 0$, then both left and right branches have subbranches on the right and left of the y-axis. This would mean that they would cross and hence, this tree is an overlapping tree. This is a contradiction. Hence $x = 0$.

□

Let $\theta = 45°$. Find r such that $T(r, \theta)$ is self-contacting.

First, find the word closest to the y-axis (the word L... that goes furthest to the right). Solve for r and set the x-coordinate to 0.

Such a word must then be of the form $LRRR(LR|RL)^\infty$.

$$\begin{aligned}
\text{x-coord} &= \overbrace{-r\sin(45°)}^{L} + \overbrace{r^2\sin(90°)}^{R} + \overbrace{r^3\sin(45°)}^{R} + \overbrace{r^4\sin(90°) + r^5\sin(45°)}^{RL} + ... \\
&= -\frac{r}{\sqrt{2}} + \frac{r^3}{\sqrt{2}} + r^4 + \frac{r^5}{\sqrt{2}} + ... \\
&= -\frac{r}{\sqrt{2}} + (\frac{r^3}{\sqrt{2}} + r^4)(1 + r^2 + r^4 + r^6 + ...) \\
&= -\frac{r}{\sqrt{2}} + (\frac{r^3}{\sqrt{2}} + r^4)\frac{1}{1-r^2} = 0 \\
\implies 0 &= -\frac{r(1-r^2)}{\sqrt{2}} + \frac{r^3}{\sqrt{2}} + r^4 \\
\implies 0 &= \underbrace{r(-1 + r^2 + r^2 + \sqrt{2}r^3)}_{\text{has solutions } r=0,0.593...,1.003...\pm(0.4287...)i}
\end{aligned}$$

#47

Hence $T(45°, 0.593...)$ is the desired self-contacting tree. Any other $LRRR(LR|RL)^\infty$ will give the same value for r, and the set of self-contacting points on the y-axis form a Cantor-like set.

Find r such that $T(60°, r)$ is a self-contacting tree.

Again, we first find the point on the left branch with x-coordinate furthest to the right, which is $LRRR(LR)^\infty$.

$$\begin{aligned}
\text{x-coord} &= \overbrace{-r\sin(60°)}^{L} + \overbrace{r^2\sin(0°)}^{R} + \overbrace{r^3\sin(120°)}^{R} + \overbrace{r^4\sin(60°) + r^5\sin(120°)}^{RL} + ... \\
&= -r\sin(60°) + r^3\sin(60°) + r^4\sin(60°) + r^5\sin(60°) + ... \\
&= \sin(60°)(-r + r^3 + r^4 + r^5 + ...) \\
&= \sin(60°)(-r + \frac{r^3}{1-r}) = 0 \\
\implies 0 &= -r + \frac{r^3}{1-r} \\
\implies 0 &= r^2 - r + r^3 \\
\implies 0 &= \underbrace{r(r^2 + r - 1)}_{r=0,0.6181...,-1.6181...}
\end{aligned}$$

#48

The value we want is $r = 0.6181...$

Let $0 < \theta \leq 90°$. Find a formula for r such that $T(r, \theta)$ is self-contacting.

We need to find N such that $(N-1)\theta \leq 90° \leq N\theta$.

After $N-1$ right turns, we are heading right, maybe slightly up. After N right turns, we are heading to the right and slightly down.

This gives us the word $LRR^N(LR)^\infty$ which has maximal x-coordinate when starting from the left branch.

Aside : If θ divides $90°$, then $(N-1)\theta \leq 90° \leq N\theta$ does not have a unique solution which leads to an infinite number of self-contacting points. #49

$$\text{x-coord} = -r\sin\theta + r^3\sin\theta + r^4\sin 2\theta + ... + r^{N+1}\sin(N-1)\theta + r^{N+2}\sin N\theta + r^{N+3}\sin(N-1)\theta + ... = 0$$

$$\implies 0 = -r\sin\theta + \sum_{i=1}^{N-2}\sin(i\theta)^{i+2} + \frac{r^{N+1}\sin(N-1)\theta + r^{N+2}\sin N\theta}{1-r^2} \quad (*)$$

which has a unique solution between 0 and 1.

Theorem 6.3. *If $0 < \theta \leq 90°$ and r satifies (*) then $T(r, \theta)$ is self-contacting.*

6.2 Space-filling trees

The formula (*) we obtained works fine for $\theta = 90°$ and the r-value that gives a self-contacting tree is $0 = -r + r^3 + r^5 + r^7 + ... = -r + \frac{r^3}{1-r^2} \implies -r + 2r^3 = 0$ which has three solutions, $r = 0, \pm\frac{1}{\sqrt{2}}$. Only $r = \frac{1}{\sqrt{2}}$ make sense. So the tree $T(\frac{1}{\sqrt{2}}, 90°)$ is self-contacting.

This tree has other special properties we now wish to explore. If we add an infinite number of branches, this tree is a solid rectangle.

Definition 6.5 (Space-filling)**.** *We say a tree is space filling if there exists a region such that every point in the region is in the tree.*

Theorem 6.4. *$T(\frac{1}{\sqrt{2}}, 90°)$ is space-filling.*

Proof. Consider a random point $a_1a_2a_3..., a_i \in \{L, R\}$. What are the possible y-values of this point? The value can take $1 \pm \frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm ...$, which can be any value between 0 and 2.

What about the possible x-values? The value can take $\sqrt{2}(\pm\frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm ...)$, which can be any value between $-\sqrt{2}$ to $\sqrt{2}$.

The key thing to notice is that the x-coordinate is determined by $a_1, a_3, a_5, ...$ and if they go to the left or right. The y-coordinate is determined by $a_2, a_4, a_6, ...$ and if they go up or down. The two are independent. Hence we have a solid rectangle.

Note : If this point is on one of the branches, then this point is not unique. If the point is not on the branch, it will be unique.

□

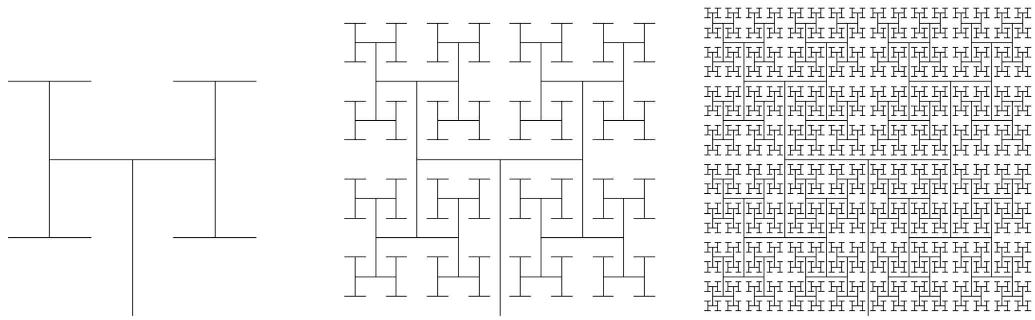


Figure 10: Space filling tree at $\theta = 90$ for various iterations.

Theorem 6.5. *$T(\frac{1}{\sqrt{2}}, 135°)$ is space-filling.*

The triangle that is filled in is bounded by the points $(0,0),(1,1)$ and $(-1,1)$.

Proof. Notice when we add the first branch from $(0,0)$ to $(0,1)$, we divide this triangle into two equal sized triangles of half the size. When we add the next two branches L and R, we divide each of these triangles into two equal-sized triangles of half the size again. That is, four triangles for $\frac{1}{4}$ the size.

As these triangles get arbitrarily small, we can get arbitrarily close to any point in the triangle. This proves the result.

□

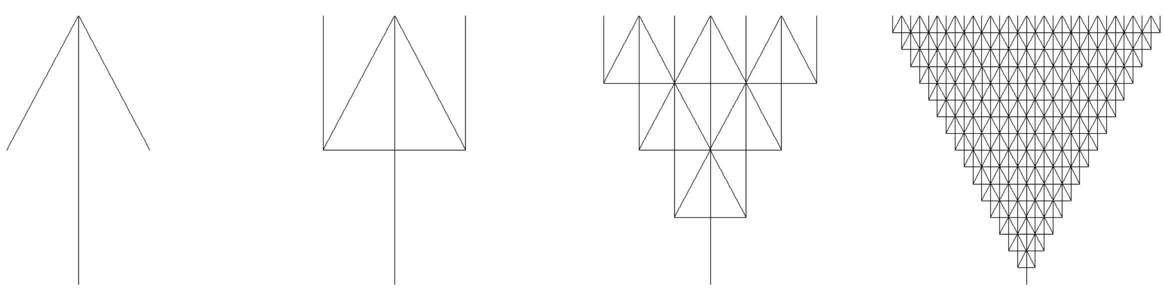


Figure 11: Space filling tree at $\theta = 135$ for various iterations.

Theorem 6.6. *If $135° \leq \theta < 180°$, then the point on the left branch furthest to the right is LL.*

Corollary 6.1. *The value of r that achieves self-contacting satisfies $0 = -r\sin\theta - r^2 \sin 2\theta$ or equivalently, $r = -\frac{\sin\theta}{\sin 2\theta}$.*

6.3 Dimension

What is the dimension of $T(r,\theta)$, assuming that it is not overlapping?

Let $T(r,\theta)$ be a tree. We will consider $N_T(r^k)$ for various k. Recall that $N_T(r^k)$ is the number of $r^k \times r^k$ boxes needed to cover T.

Let k be sufficiently large so that 1) $N_T(r^k)$ is large and 2) r^k is small. Let $M = N_T(r^k)$. What is $N_T(r^{k+1})$?

As T needs M boxes of size r^k to be covered, we see that the left branch needs M boxes of size r^{k+1} as the left branch is a copy of T, scaled by r. Similarly, we need M boxes of size r^{k+1} to cover the right branch. Finally, we will need $\approx \frac{1}{r^{k+1}}$ boxes to cover the trunk from $(0,0)$ to $(0,1)$. Hence we have that :

$$N_T(r^k) = M \qquad\qquad N_T(r^{k+1}) = 2M + \frac{1}{r^{k+1}}$$

$$\begin{aligned} N_T(r^{k+2}) &= 2N_T(r^{k+1}) + \frac{1}{r^{k+2}} \\ &= 2(2M + \frac{1}{r^{k+1}}) + \frac{1}{r^{k+2}} \end{aligned}$$

Case 1 : $r < \frac{1}{2}$

$$\begin{aligned} N_T(r^{k+2}) &= 2^2M + \frac{2}{r^{k+1}} + \frac{1}{r^{k+2}} \\ N_T(r^{k+3}) &= 2^3M + \frac{4}{r^{k+1}} + \frac{2}{r^{k+2}} + \frac{1}{r^{k+3}} \qquad (1) \\ &= 2^3M + \frac{1}{r^{k+3}}(1 + 2r + 4r^2) \\ &\dots \\ N_T(r^{k+n}) &= 2^nM + \frac{1}{r^{k+n}}(1 + 2r + (2r)^2 + \dots + (2r)^{n-1}) \end{aligned}$$

We see that $2r < 1$ as $r < \frac{1}{2}$. Hence $(1 + 2r + (2r)^2 + \dots + (2r)^{n-1}) < \frac{1}{1-2r}$ and hence, this sum is bounded by a constant as $n \to \infty$, say C. So the dimension satisfies

$$\lim_{n\to\infty} \frac{\log N_T(r^{k+n})}{\log \frac{1}{r^{k+n}}} = \lim_{n\to\infty} \frac{\log(2^nM + \frac{1}{r^{k+n}}C)}{\log(\frac{1}{r^{k+n}})}$$

Because $r < \frac{1}{2}$, we have $\frac{1}{r} > 2$. Hence, the second term dominates as $n \to \infty$.

$$\lim_{n\to\infty} \frac{\log(2^nM + \frac{1}{r^{k+n}}C)}{\log(\frac{1}{r^{k+n}})} = \lim_{n\to\infty} \frac{\log(\frac{1}{r^{k+n}}C)}{\log(\frac{1}{r^{k+n}})} = \lim_{n\to\infty} \frac{\log(\frac{1}{r^{k+n}}) + \log C}{\log(\frac{1}{r^{k+n}})} = 1$$

Case 2 : $r > \frac{1}{2}$

Following (1), we rewrite $N_T(r^{k+n})$ differently:

$$\begin{aligned}
N_T(r^{k+3}) &= 2^3 M + \frac{4}{r^{k+1}} + \frac{2}{r^{k+2}} + \frac{1}{r^{k+3}} \\
&\dots \\
N_T(r^{k+n}) &= 2^n M + \frac{1}{r^{k+n}} + \frac{2}{r^{k+n-1}} + \dots + \frac{2^{n-1}}{r^{k+1}} \\
&= 2^n M + 2^{k+n}\left(\frac{1}{r^{k+n}2^{k+n}} + \frac{1}{r^{k+n-1}2^{k+n-1}} + \dots + \frac{1}{r^{k+1}2^{k+1}}\right) \\
&= 2^n M + 2^{k+n}\underbrace{\left(\frac{1}{(2r)^{k+n}} + \frac{1}{(2r)^{k+n-1}} + \dots + \frac{1}{(2r)^{k+1}}\right)}_{C}
\end{aligned}$$

Note that as $r > \frac{1}{2}$, $2r > 1$ hence $\frac{1}{2r} < 1$. This last term is bounded by a constant C. So we have that

$$\dim = \lim_{n\to\infty} \frac{\log N_T(r^{k+n})}{\log \frac{1}{r^{k+n}}} = \lim_{n\to\infty} \frac{\log(2^n M + 2^{k+n} C)}{\log \frac{1}{r^{k+n}}} = \lim_{n\to\infty} \frac{\log(2^{k+n}) + \log(C)}{\log \frac{1}{r^{k+n}}} = \frac{\log 2}{\log \frac{1}{r}}$$

Note : Recall $T(\frac{1}{\sqrt{2}}, 90°), T(\frac{1}{\sqrt{2}}, 135°)$. By this formula, they would have dimension $\frac{\log 2}{\log \sqrt{2}} = 2$ as expected.

7 Non-Linear Systems & The Henon Attractor

7.1 Non-Linear Systems

Recall, when we looked at a linear system $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(\vec{x}) = A\vec{x} + \vec{b}$, that the properties of the fixed point depend upon the eigenvalues of the matrix A.

Similar things can be done for non-linear systems. For convenience, we will restrict our attention to $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Find the fixed points of $f(x, y) = (\frac{x^2+y^2}{2}, xy)$ and classify them.

The fixed points occur when

$$x = \frac{x^2 + y^2}{2} \quad y = xy$$
$$\implies x = 1, y = \pm 1 \quad \text{or} \quad y = 0, x = 2 \text{ or } 0$$

So the fixed points are (1, 1), (1, -1), (0, 0), (2, 0). How do we find if they are attractive/repelling?

Close enough to a point (x_0, y_0), we have that $f(x, y)$ is approximately linear, so we use a linear approximation matrix.

$$\text{i.e. } f(x,y) = f(x_0, y_0) + \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \text{ where } f(x, y) = (f_1(x, y), f_2(x, y))$$ #50

In this case, this becomes $M = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$.

At (0, 0), $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has both eigenvalues 0 and the eigenvectors span the space. This fixed point is attractive.

At (2, 0), $M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ has both eigenvalues 2. This is repelling.

At (1, 1), $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues 0, 2. This point is neither attractive or repelling.

At (1, -1), $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has eigenvalues 0, 2, same thing.

Recall : Let A be a closed set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We defined $d(x, A) = \min_{y \in A} |x - y|$.

Definition 7.1 (Attractor). *We say a closed set $A \subseteq \mathbb{R}^n$ is an attractor for $f : \mathbb{R}^n \to \mathbb{R}^n$ if $f(A) = A$ and there exists some $\epsilon > 0$ such that if $d(x, A) < \epsilon$, then $\lim_{n\to\infty} d(F^{[n]}(x), A) = 0$.*

Thus, points close to attractors get closer to the attractor upon iteration, but need not settle on a specific point.

An attractive fixed point a is an attractor. Two attractive fixed points or an attractive periodic cycle or union of such cycles are also attractors.	#51

Definition 7.2 (Repellor)**.** *We say a closed set A is a repellor if $f(A) = A$ and there exists $\epsilon > 0$ such that if $0 < d(x, A) < \epsilon$, then $d(x, A) < d(f(x), A)$.*

Repelling fixed points, cycles or union thereof are repellors. The Cantor Set is a repellor for $T_{\frac{3}{2}} = \begin{cases} 3x & x < \frac{1}{2} \\ 3 - 3x & x \geq \frac{1}{2} \end{cases}$. The Julia Set J_c for $g_c(z) = z^2 + c$ is a repellor.	#52

7.2 Henon Map

Definition 7.3 (Henon Map)**.** *The Henon map is defined as* $H_{a,b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$.

Claim : Fr the correct values of a and b, this function has an attractor with non-integer dimension.

Pictures of the Henon Map `http://en.wikipedia.org/wiki/H%C3%A9non_map`

Any "line" is actually made up of infinitely many lines and the cross section is Cantor-like with $1 < \dim < 2$.

Theorem 7.1. *If $b \neq 0$, then $H_{a,b}$ is one-to-one and invertible.*

Proof. To see that it is one-to-one, assume $H_{a,b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = H_{a,b}\left(\begin{bmatrix} v \\ w \end{bmatrix}\right)$. Then $\begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + w \\ bv \end{bmatrix}$. From the second coordinate, $bx = bv$. As $b \neq 0$, we have $x = v$. Using $x = v$, the first coordinate gives $y = w$. Hence this is one-to-one.

Assume $H_{a,b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$. We want to find $H_{a,b}^{-1}\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$. Using $\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$, solve for x, y in terms of u, w.

By the second equation, $x = \frac{w}{b}$. The first equation gives $y = v + ax^2 - 1 = v + aw^2/b - 1$.

$\therefore H_{a,b}^{-1}\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) = \begin{bmatrix} w/b \\ v + aw^2/b - 1 \end{bmatrix}$.

□

Consider the fixed points of $H_{a,b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

From the second equation, $y = bx$. Then, from the first,

$$\begin{aligned} x = 1 - ax^2 + y &= 1 - ax^2 + bx \\ \implies ax^2 - bx + x - 1 &= 0 \\ \implies ax^2 + (1-b)x - 1 &= 0 \\ \implies x &= \frac{(b-1) \pm \sqrt{(1-b)^2 + 4a}}{2a} \end{aligned}$$

This implies that if $(1-b)^2 + 4a < 0$, then we have no fixed points.
If $(1-b)^2 + 4a = 0$, then we have one fixed point.
If $(1-b)^2 + 4a > 0$, then we have exactly two fixed points.

When are the fixed points attractive or repelling? To determine this, we need to examine

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -2ax & 1 \\ b & 0 \end{bmatrix}$$

This has two eigenvalues that satisfy $\det(A - \lambda I) = (\lambda + 2ax)\lambda - b = 0$ which give

$$\lambda = \frac{-2ax \pm \sqrt{4a^2x^2 + 4b}}{2} = -ax \pm \sqrt{a^2x^2 + b}$$

Claim : If $0 < b < 1$ and $a \in (\overbrace{-\frac{1}{4}(1-b)^2}^{\text{tight bound}}, \overbrace{\frac{3}{4}(1-b)^2}^{\text{not tight bound}})$, then the fixed point $x = \frac{(b-1) \pm \sqrt{(1-b)^2+4a}}{2a}$ and $y = bx$ is attractive.

Lemma 7.2. *If a, b, x satisfy the above restrictions, then $-1 < b - 1 < \pm 2ax < 1 - b < 1$.*

Proof. As $0 < b < 1$, we get $-1 < b - 1$ and $1 - b < 1$.

$x = \frac{(b-1) \pm \sqrt{(1-b)^2+4a}}{2a} \implies 2ax = (b-1) + \underbrace{\sqrt{(b-1)^2 + 4a}}_{>0} > b - 1$. Since $2ax > b - 1$, we also get that $-2ax < 1 - b$.

We know that $a < \frac{3}{4}(b-1)^2$ by assumption. Therefore

$$\begin{aligned} 2ax &= (b-1) + \sqrt{(b-1)^2 + 4a} \\ &< (b-1) + \sqrt{(b-1)^2 + 3(b-1)^2} \\ &< (b-1) + \sqrt{4(b-1)^2} \\ &< (b-1) + |2(b-1)| \\ &= (b-1) + 2(1-b) = 1 - b \end{aligned}$$

This gives $2ax < 1 - b$ and equivalently, $-2ax > b - 1$. This completes the proof.

$\square$

We now wish to show that this fixed point is attractive. We have $-1 < b - 1 < \pm 2ax < 1 - b < 1$ from our lemma and $\lambda_- = -ax - \sqrt{a^2x^2 + b}$, $\lambda_+ = -ax + \sqrt{a^2x^2 + b}$.

Our goal is to show $-1 < \lambda_\pm < 1$. Consider the two cases $ax < 0, ax > 0$ for λ_+ (the case $ax = 0$ gives $\lambda_+ = \sqrt{b} < 1$ as required).

If $ax > 0$, $\lambda_+ = -ax + \sqrt{a^2x^2 + b} > -ax + \sqrt{a^2x^2} > -ax + ax = 0 > -1$.

If $ax < 0$, $\lambda_+ = -ax + \sqrt{a^2x^2 + b} > -ax > -\frac{1}{2} > -1$.

Now, $\lambda_+ = -ax + \sqrt{a^2x^2 + b} < -ax + \sqrt{a^2x^2 + 2ax + 1} = -ax + \sqrt{(ax + 1)^2} = -ax + ax + 1 = 1$. So $-1 < \lambda_+ < 1$ as required.

A similar proof will show $-1 < \lambda_- < 1$. This gives that the fixed point is attractive.